Separation and the resource λ-calculus

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Abstract

This is the report on the internship M2 (Master 2), second semester of the academic year in Fundamental Computer Science at "ENS de Lyon". It summarizes the studies taken place at the research group LDP - "Logique de la Programmation" - in Marseille between March and June 2011.

We will present some interesting results in logic. In particular we will briefly introduce, in the first part of this document, the pure λ-calculus focusing on the separation problem and its semantic implications. Thus, we will deal with Böhm's separation theorem and prove it in a different manner from the original one, following a technique due to Joly.

In the second part of this report, we will present the resource λ-calculus, introducing the Taylor expansion of resource λ-terms. Finally, we will present a recent result about separation in this setting comparing it, and its corollaries, to the original one.

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1 Preliminaries

In the theory of λ-calculus Böhm's theorem, also known as separation theorem, is a key result stating that a non-trivial model of the $\beta\eta$-equivalence cannot identify different $\beta\eta$-normal forms.

The original aim of the internship was to study the separation problem in a quantitative context; that is, investigating on the possibilities to extend Böhm's theorem to an extension of the λ-calculus having algebraic properties (i.e. linear combinations of λ-terms).

This modified version of the pure λ-culculus is motivated by the recent introduction of some quantitative semantics of linear logic and the discovery of correlations with concurrency theory and quantum computing. For instance, it is possible to think about sums as expressing non-determinism.

As the reader will soon understand, a slightly different λ-calculus has been object of study. Indeed, since the very first days at LDP, it has been decided to investigate and study the same problem in a qualitative context. The main reason under this decision is due to a recent scientific publication dealing with Böhm's theorem in this setting. The idea was then to make experience on separation problem in this kind of qualitative calculus in order to face the original settings in a second moment. Unfortunately, the time ran out quickly.

It has been decided to study the separation problem, firstly, in the context of the pure λ-calculus following the huge literature on the subject. Thus, problems related to the generalization of the original theorem have been taken into account. Moreover, it has taken a look to thematics regarding the problem of solvability which is a notion and a problem closed to that of separation. In a second moment, it has been studied the resource λ-calculus, a recent extension of the pure λ-calculus, trying to figure out how, and in which sense, this calculus is related to the differential λ-calculus. Finally, it has been worked on the understanding of the Böhm's theorem for the resource λ-calculus due to Manzonetto and Pagani. It turned out to be a quite difficult task because of its strong technical characterization.

2 Originally, the *pure* $\boldsymbol{\lambda}$-calculus

2.1 Introduction

The (pure) λ-calculus [1] was introduced around 1930 by Alonzo Church. From the beginning, the calculus was part of series of studies intended to develop a general theory of functions. At the same time, a second aim was to extend that theory with logical notions to provide a foundation for logic and mathematics.

The latter point failed. Church's original system was inconsistent as shown by Kleene and Rosser through a construction of a paradox. Then, many studies were made to provide a consistent theory. An example was Curry's pure combinatory logic, but it turned out that the logical part of the theory was too weak to be used as a foundation.

After the discovery of the paradox, Church left his foundational program and gave a consistent subtheory of his original system dealing only with the first aim, the λI-calculus.

The developed theory turned out to be quite successful and useful. In fact, based on the expressiveness of this calculus, Church proposed a formalization of the notion of "effectively computable" by the concept of λ-definability. In the meantime, Turing gave an analysis of the functions computable by machines and showed that the resulting notion is equivalent to the λ-definability. The result is well known as the Church-Turing thesis.

In this report we are interested in a general form of the λI-calculus, namely the λK-calculus, also simply known as λ-calculus.

The rise of computers carried an extensive development of programming languages and the computer science in general. It turned out that the λ-calculus was a natural mathematical and formal object suitable for studying the foundations of programming languages, in particular those following the functional paradigm. Indeed, the λ-calculus has been widespread used in specification of programming languages features, in language design and implementation, and in the study of type systems.

2.2 Basics

Syntax. The λ-calculus [1] is a formal system of computational functions, based on two basic operations: λ-abstraction and application.

Functions are first-class objects in this calculus. Roughly speaking, a function is written in the form $\lambda x.M$ where the abstraction remarks that the function is parametrized in x[1] and M is a λ-term constituting the function's body. A λ-term is either a variable, a function or an application of two λ-terms $(M_1)M_2$.

Definition 2.1. The set of λ-terms Λ is the smallest set generated by the following grammar:

$$M, N ::= x \mid \lambda x.M \mid (M)N$$

where x is an arbitrary variable.

In λ-terms of the form $\lambda x.M$, the λ-abstraction binds the variable x in the body M of the function, which is also known as the scope of the binder. An occurrence of a variable x is bound whenever it is under a binder for x, free otherwise. The set of free variables of a λ-term M, denoted $FV(M)$, is the set of variables with free occurrences in M. We will say that a λ-term M is closed if $FV(M) = \emptyset$, and we will call it a combinator.

Let us consider the following example of a λ-term T: $x(\lambda x.(\lambda x.yx)x)$[2]. It is easy to check that $FV(T) = \{x, y\}$, in fact there is no binder for y and the first occurrence of x is free. Obviously, T is not a combinator.

Note that in an abstraction $\lambda x.M$, the use of the particular variable x is no longer relevant. It is just a way to denote the places in which the parameter of the function built by abstraction occurs. Therefore the variable x could be replaced by another one y, provided that this would not cause that some occurrences of y in M be improperly associated to the "renamed" abstraction. For example, in $\lambda x.xy$ the x can be replaced by the variable z to form the λ-term $\lambda z.zy$, but not by the variable y that would lead to $\lambda y.yy$, where the original y would be "captured". The equivalence induced by the variable replacing process is known as α-equivalence, denoted $\equiv_\alpha$. From now on, we will consider λ-terms up to α-equivalence.

Operational semantic. Now we will give the computational behavior of the calculus in terms of reduction rules.

Definition 2.2. A relation $\mathtt{R}$ on Λ is *compatible* if it satisfies the following conditions for all $M, N, Z \in \Lambda$.

(1) If $M \mathrel{\mathtt{R}} N$ then $\lambda x.M \mathrel{\mathtt{R}} \lambda x.N$, for all variables x.

(2) If $M \mathrel{\mathtt{R}} N$ then $MZ \mathrel{\mathtt{R}} NZ$ and $ZM \mathrel{\mathtt{R}} ZN$.

The main reduction notion is called β-reduction and it is the least compatible relation $\to_\beta$ on Λ satisfying the following rule:

$$(\lambda x.M)N \to_\beta M[x := N]$$

where $M[x := N]$ denotes the λ-term obtained substituting each free occurrence of x in M by the argument N. Here, we maybe need to make some renaming to avoid free variables in N to be captured by some λ-abstractions in M. The rule can be seen as the process of instantiation of a formal parameter to an actual one at the moment of a function application to an argument.

A λ-term of the form $(\lambda x.M)N$ is called β-redex and the λ-term $M[x := N]$ is said to arise by contracting the redex. We will say that a λ-term M is in β-normal form if there is no M' such that $M \to_\beta M'$, i.e. M does not contain a β-redex. We will denote NF_β the set of λ-terms in normal form.

Lemma 2.3. *Normal forms are of the shape*[3] $\vec{\lambda y}_m.x\vec{M}_n$ *where* $m, n \geq 0$ *and all* $M_i \in NF_\beta$*. The variable* x *is called the head variable of the* λ*-term.*

Definition 2.4. *Head normal forms* are of the shape $\vec{\lambda x}_m.x\vec{M}_n$, where $m, n \geq 0$. In a λ-term of the form $T \triangleq \vec{\lambda x}_m.(\lambda x.N)M_0\vec{M}_n$, where $m, n \geq 0$, the λ-term $(\lambda x.N)M_0$ is called the *head redex* of T.

We will call multi-step β-reduction the transitive and reflexive closure of $\to_\beta$, denoting it $\to_\beta^*$. Furthermore, we will define the congruence $\equiv_\beta$ on Λ, known as β-equivalence, as the least equivalence relation containing $\to_\beta$.

[1] The reader can associate to x the notion of formal parameter typical of definitions of functions in programming languages like OCaml, Lisp, etc.

[2] Notice that we should have written $x(\lambda x.(\lambda x.(y)x)x)$ according to the grammar. But as in this case, we will let us write λ-terms without parenthesis when the sense of the application is obvious.

[3] We will denote $\vec{a}_n$ any n-sequence of a_i objects, $1 \leq i \leq n$; namely $\vec{a}_n \triangleq a_1 \ldots a_n$, where $n \in \mathcal{N}$. Note that the objects could be different from each other, but they are of the same sort. In particular, $\vec{\lambda x}_n \triangleq \lambda x_1 \ldots \lambda x_n$.

Even if so syntactically simple, in the λ-calculus it is possible to write terms with infinite computation. A typical example is $\mathbf{\Omega}$ which does not have a NF$_\beta$:

$$\mathbf{\Omega} \to_\beta \mathbf{\Omega} \to_\beta \mathbf{\Omega} \to_\beta \dots$$

where $\mathbf{\Omega} \triangleq \boldsymbol{\delta\delta} \triangleq (\lambda x.xx)(\lambda x.xx)$.

Another reduction notion is the η-reduction that permits to take into account the extensional properties of the calculus. The η-reduction is defined as the least compatible relation $\to_\eta$ on Λ satisfying the following rule:

$$\lambda x.Mx \to_\eta M, \text{ if } x \notin FV(M)$$

The rule encodes the intuition which wants two λ-terms $\lambda x.Mx$ (where $x \notin FV(M)$) and M to be the same because of the result they yield when applied to a λ-term N, that is MN.

As above, we will consider a multi-step η-reduction, denoting it $\to_\eta^*$, and the congruence $\equiv_\eta$ known as η-equivalence.

Consider the following λ-term T, where its β-redexes are underlined:

$$\underline{(\lambda x\lambda y.y)\underline{\underline{(\boldsymbol{\delta\delta})}}}$$

Without entering into a discussion about reduction strategies, it is interesting to know if such a λ-term can produce only one result; that is, if contracting any of its β-redexes the resulting λ-term is the same. The following theorem explains why it is the case and why the λ-calculus is said to be confluent.

Theorem 2.5 (**Confluence**). *Let $M \in \Lambda$. If $M \to_\beta^* P$ and $M \to_\beta^* Q$, then there is a $N \in \Lambda$ with $P \to_\beta^* N$ and $Q \to_\beta^* N$.*

The reader should note that the above result says nothing about the termination property of the calculus. It has just shown previously that λ-calculus admits non-terminating computations.

In practice the theorem ensures that, no matter how the contracting sequence of β-redexes is built, if a term has a normal form then it can be eventually reached, even in presence of possible infinite reductions.

For example, in the last term proposed, it is possible to build an infinite reduction chain by contracting always the β-redex on the right (i.e., $(\boldsymbol{\delta\delta})$). But at the same time, it is always possible to contract the β-redex on the left, obtaining the normal form $\lambda y.y$.

2.3 Böhm's theorem

In this section we will present one of the most important theorems regarding the theory of λ-calculus. The theorem is due to Corrado Böhm [2], an Italian mathematician.

It is one of those many theorems bringing a deep semantic result but developed in a syntactical fashion. This statement will be clarified throughout the section.

Intuitively, being the results of the evaluation of a λ-term, when this computation has an end, normal forms can be seen as values and Böhm's theorem ensures that two values are equal if and only if they are written in the same way. The theorem states that if two normal λ-terms are distinct then it can be built a context[4] to separate them. Indeed, this theorem is also known as the *separation* theorem for the pure λ-calculus.

Moreover, the way in which it is possible to separate two distinct λ-terms is internal to the calculus; that is, the combinatorial operations of separation are expressed by means of suitable λ-terms.

Theorem 2.6 (**Böhm**). *Let M, N be two closed normal λ-terms, which are not $\beta\eta$-equivalent. Then there exist closed λ-terms $P_1, \dots, P_k$ such that:*

$$(M)P_1 \dots P_k \equiv_\beta \mathbf{F}, \textit{ and}$$
$$(N)P_1 \dots P_k \equiv_\beta \mathbf{T}$$

where the combinators $\mathbf{T}$, $\mathbf{F}$ are respectively $\lambda x\lambda y.x$ and $\lambda x\lambda y.y$.

Corollary 2.7. *Let M, N be two closed normal λ-terms, which are not $\beta\eta$-equivalent, and V, W two arbitrary λ-terms. Then there exist closed λ-terms $P_1, \dots, P_k$ such that:*

$$(M)P_1 \dots P_k \equiv_\beta V, \textit{ and}$$
$$(N)P_1 \dots P_k \equiv_\beta W$$

[4]A context is a λ-term with a hole used as a placeholder for an actual λ-term.

Proof. By the theorem 2.6, it holds $(M)P_1 \dots P_k \equiv_\beta \mathbf{F}$ and $(N)P_1 \dots P_k \equiv_\beta \mathbf{T}$; thus it follows:

$$(M)P_1 \dots P_k WV \equiv_\beta \mathbf{F}WV \equiv_\beta V, \text{ and}$$
$$(N)P_1 \dots P_k WV \equiv_\beta \mathbf{T}WV \equiv_\beta W$$

□

It follows that the equational theory induced by $\beta\eta$-reductions is complete for its normal forms. In fact, trying to equate any pair of non $\beta\eta$-equivalent normal λ-terms would correspond to equating the whole set $\mathrm{NF}_{\beta\eta}$, forcing its collapse into a single point.

In other words, the $\beta\eta$-equivalence is maximal among the λ-compatible equivalence relations on Λ containing the β-equivalence.

Corollary 2.8. *Let $\simeq$ be an equivalence relation on Λ containing $\equiv_\beta$, such that holds $M \simeq N \Rightarrow (M)U \simeq (N)U$ and $M \simeq N \Rightarrow \lambda x.M \simeq \lambda x.N$ for every λ-term T, T', U and every variable x. If there exist two normalizable non-$\beta\eta$-equivalent λ-terms V, W such that $V \simeq W$, then $Z \simeq Z'$ for all λ-terms Z, Z'.*

Proof. Let V', W' be a closure of V, W. They are in the form $\lambda\vec{x}.V$ and $\lambda\vec{x}.W$. By definition, $V' \simeq W'$ but $V' \not\equiv_{\beta\eta} W'$. Thus, by corollary 2.7 it follows that $(V')P_1 \dots P_k \equiv_\beta Z$ and $(W')P_1 \dots P_k \equiv_\beta Z'$. Therefore $Z \simeq Z'$. □

Böhm-out technique. As in almost all the cases of relevant results in mathematics, Böhm's theorem is interesting not only in the statement that it asserts, but probably, mostly in the constructions needed by its proof; in particular the so called Böhm-out technique. At the base of this technique there is the deep understanding of the structural properties of λ-terms and how to exploit them by the computational mechanism behind the β-rule.

In the current paragraph, we will only sketch the original proof of the theorem, mainly focusing on the general ideas and the analysis of the Böhm-out technique. For a complete proof, but different from the original one, look at section 2.4.

The following type of λ-terms have a crucial role in the Böhm-out technique. Roughly speaking, they are useful to reorganize in a particular fashion a sequence of λ-terms to which they are applied to.

Definition 2.9. A *n-permutator*, denoted $\mathbf{P}_n$, is a λ-term of the form:

$$\lambda\vec{x}_n \lambda x_{n+1}.x_{n+1}\vec{x}_n$$

To remark and study the structural properties of λ-terms, Barendregt introduced the notion of Böhm tree of a λ-term.

Definition 2.10. The *Böhm tree* [1, §10] of a λ-term M, indicated with $BT(M)$, is defined as follows:

$$BT(M) = \begin{cases} \bot & \text{if } M \text{ has no head normal form} \\ \begin{array}{c} \lambda\vec{x}_n.y \\ \swarrow \quad \downarrow \quad \searrow \\ BT(M_1) \quad \cdots \quad BT(M_m) \end{array} & \text{if } M \equiv_\beta \lambda\vec{x}_n.y\vec{M}_m \end{cases}$$

Definition 2.11. Given $M, N \in \Lambda$, if $BT(M) \neq BT(N)$ then it is said that M *differs* from N.

In particular, given two λ-terms, it is possible to analyze where they differ.

Definition 2.12. Given $M, N \in \Lambda$ of the form $\lambda\vec{x}_n.x\vec{M}_p$ and $\lambda\vec{y}_m.y\vec{N}_q$ respectively, it is said that M differs from N at the *top level* if one of the following holds:

(1) $n \neq m$ (2) $x \neq y$ (3) $p \neq q$

otherwise, they differ at a *lower level*; namely, $\exists i$ s.t. $BT(M_i) \neq BT(N_i)$.

Proof (sketch) of 2.6. The proof reasons by induction on the depth level of the Böhm trees where the two given λ-terms differ. Intuitively, if they differ at the top level is quite easy to build a list of arguments to force the computation ends with $\mathbf{T}$ and $\mathbf{F}$ as results; otherwise, the Böhm-out technique is used to extract the difference, bringing it at an upper level and, eventually, to the top level.

Although the top level is the base case of the induction, it is not particularly difficult to figure it out how to cope with it. On the contrary, the induction case is not so obvious and it is the key technical point of the proof. Remembering that Böhm's theorem takes into account only closed λ-terms in normal form, let's consider the following scenario with no differences at the top level, $\lambda\vec{x}_n.x_i\vec{M}_m$ and $\lambda\vec{x}_n.x_i\vec{N}_m$, and assume we have to extract the difference from the subterms M_k, N_k. Two possibilities arise:

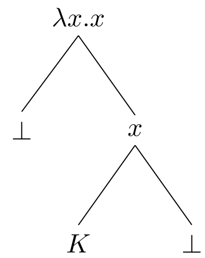
1. If the head variable x_i has no occurrences into the path who brings to the different subtrees contained in the the λ-terms M_k, N_k, then all we need is to instantiate x_i with the appropriate projection, i.e. the λ-term $\lambda y_1 \dots \lambda y_m.y_k$, to move the difference to an upper level;

2. Otherwise, we have to use a more sophisticated technique. In fact, instantiating all the x_i with a projection may pull out some subtrees on the path to the difference but, at the end, would lead to a structural modification of the λ-terms without any advantages on the extraction process.

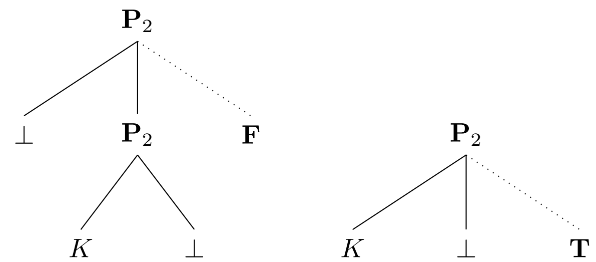
   Let h be the maximal number of children of x_i in the path leading to the difference. The idea is to instantiate all the x_i with $\mathbf{P}_h$ and build the rest of the context, to which the two λ-terms are applied to, following the process based on projections. Indeed, notice that after the substitution $[x_i := \mathbf{P}_h]$ the only possible scenario is of the first type.

□

The following example explains how a n-permutator works and its need in the proof of Böhm's theorem. Let us consider the λ-term $T \triangleq \lambda x.x\mathbf{\Omega}(xK\mathbf{\Omega})$. Below, its Böhm tree is drawn, and suppose we want to extract K.



One could think to apply T to the combinator $\mathbf{F}$ to pull out, at first, the second child of the root. The problem is the presence of a second instance x to the path leading to the λ-term K. Indeed, the result would be $(T)\mathbf{F} \to_\beta^* \mathbf{\Omega}$. Now take the λ-term $\mathbf{P}_2\mathbf{F}\mathbf{T}$ and consider the following two figures regarding the two situations where $\mathbf{P}_2$ acts.



Roughly speaking, when $\mathbf{P}_2$ is the root, it is like adding a new child to its node with the first λ-term of the list of parameters. Hence, by some reductions, it takes this parameter to the root to eventually select the right subtree to pull out. Moreover, doing so it does not mess up the structure of subtrees. The reduction process is $(T)\mathbf{P}_2\mathbf{F}\mathbf{T} \to_\beta \mathbf{P}_2\mathbf{\Omega}(\mathbf{P}_2K\mathbf{\Omega})\mathbf{F}\mathbf{T} \to_\beta \mathbf{F}\mathbf{\Omega}(\mathbf{P}_2K\mathbf{\Omega})\mathbf{T} \to_\beta \mathbf{P}_2K\mathbf{\Omega}\mathbf{T} \to_\beta \mathbf{T}K\mathbf{\Omega} \to_\beta K$.

2.4 Böhm's theorem à la Joly

We will now present a particular proof of Böhm's theorem due to Thierry Joly [9]. It is an interesting version thanks to its concision and its mechanical nature easily expressible in an algorithmic implementation.

Roughly speaking, Joly uses a clever technique based on nested "pair" applied to particular combinators used to select the right component of the pair at each computational step. Intuitively, the left component of the pair contains either the actual λ-terms act to separate (base case) or a pair itself (inductive case), whereas the right component contains a λ-term acting as projection to bring at the top level a subterm of the initial one.

The adoption of pairs is to avoid messing up the structure of the two λ-terms to separate. Indeed, the technique does not directly replace variables with projections, but with pairs containing the actual extraction process codified inside, specific for each level of the Böhm tree of the two λ-terms. A proper external sequence of combinators is used to select the necessary pair, hence triggering the right extraction process.

Let $\mathbf{f} \triangleq \lambda x \lambda y.y$, $\mathbf{t} \triangleq \lambda x \lambda y.x$ and let us write $T\mathbf{t}^n$ for the λ-term $T\mathbf{t}\ldots\mathbf{t}$, where the number of $\mathbf{t}$ is n.

Definition 2.13. Let $n \in \mathcal{N}$. The caption $\langle T, T' \rangle_n$ denotes the λ-term $\lambda\vec{x}_n \lambda y.y(T\vec{x}_n)(T'\vec{x}_n)$, where $\vec{x}_n \triangleq x_1 \ldots x_n$ and y are fresh variables.

We have clearly that $\langle T, T' \rangle_{n+1} U \to_\beta \langle TU, T'U \rangle_n$. Moreover, we will be able to extract each component by $\langle T, T' \rangle_0 \mathbf{t} \to_\beta T$ and $\langle T, T' \rangle_0 \mathbf{f} \to_\beta T'$.

Lemma 2.14. *If $M[x := N] \to^*_\beta y$ and x does not occur free in N, then for all sufficiently large $k \in \mathcal{N}$ and all λ-terms N', there are $l, l' \in \mathcal{N}$ such that $M[x := \langle N, N' \rangle_k]\mathbf{t}^l \to^*_\beta y\mathbf{t}^{l'}$.*

Proof. The proof proceeds by induction on the length of the head reduction[5] $M[x := N] \to^*_h y$.

Base cases.

- If $M \triangleq y$, then the statement obviously holds.
- If $M \triangleq x$, then $N \to^*_\beta y$ by hypothesis. But then, for all $k \in \mathcal{N}, N' \in \Lambda$ $\langle N, N' \rangle_k \mathbf{t}^{k+1} \to^*_\beta \langle N\mathbf{t}^k, N'\mathbf{t}^k \rangle_0 \mathbf{t} \to_\beta N\mathbf{t}^k \to^*_\beta y\mathbf{t}^k$.
  We obtain $l = k+1$ and $l' = k$.

Inductive cases.

- If M is not in head normal form, then $M \to_h M'$ and so $M[x := N] \to_h M'[x := N]$ by lemma 8.3.12 in [1, §8.3], telling us that the head reduction is stable under substitution. For all sufficiently large $k \in \mathcal{N}, N' \in \Lambda$, there are by induction hypothesis $l, l' \in \mathcal{N}$ such that $M'[x := \langle N, N' \rangle_k]\mathbf{t}^l \to^*_\beta y\mathbf{t}^{l'}$. It easily follows that $M[x := \langle N, N' \rangle_k]\mathbf{t}^l \to_h M'[x := \langle N, N' \rangle_k]\mathbf{t}^l \to^*_\beta y\mathbf{t}^{l'}$.
- If $M \triangleq x\vec{P}$, with $\vec{P} \triangleq P_1 \ldots P_n$ $(n \geq 1)$, then $M' \triangleq N\vec{P}[x := N]$ is not in head normal form. In fact, since $x \notin FV(N)$ it follows that $N \not\triangleq x$. Moreover, from the hypothesis $M' \to^*_\beta y$ we derive that $N \not\triangleq y$ too. Thus, $M' \to_h \to^*_\beta y$.

  For all sufficiently large $k \in \mathcal{N}$, $N' \in \Lambda$, there are by induction hypothesis $l, l' \in \mathcal{N}$ such that $N\vec{P}'\mathbf{t}^l \to^*_\beta y\mathbf{t}^{l'}$, where $\vec{P}' \triangleq \vec{P}[x := \langle N, N' \rangle_k]$. Hence, we obtain that for all $k \geq n$, $M[x := \langle N, N' \rangle_k]\mathbf{t}^{k+l+1} \triangleq \langle N, N' \rangle_k \vec{P}'\mathbf{t}^{k+l+1} \to^*_\beta \langle N\vec{P}'\mathbf{t}^{k-n}, \ldots \rangle_0 \mathbf{t}^{n+l+1} \to_\beta N\vec{P}'\mathbf{t}^{l+k} \to^*_\beta y\mathbf{t}^{l'+k}$.

□

Theorem 2.15 (Böhm). *Let T_1, T_2 be two normal λ-term, non-$\beta\eta$-equivalent, whose free variables are among $\vec{x}_r$. Then there are $\vec{U}_r$ and $\vec{V}$ with free variables in $\{y_1, y_2\}$, where $y_1 \not\triangleq y_2$, $\{x_1, \ldots x_r\} \cap \{y_1, y_2\} = \emptyset$, such that $T_i[\vec{x}_r := \vec{U}_r]\vec{V} \to^*_\beta y_i$ $(i = 1, 2)$.*

Proof. Let us firstly introduce the notion of *lexical length* of a λ-term T, denoted by $llg(T)$, defined inductively as:

$$llg(x) \triangleq 1 \qquad llg(TT') \triangleq llg(T) + llg(T') \qquad llg(\lambda x.T) \triangleq llg(T) + 2$$

[5] The β-reduction on head redexes.

The proof reasons by induction on $llg(T_1) + llg(T_2)$ and case analysis on the structure of T_1, T_2.

Base cases.

- If $T_i \triangleq x_{s_i} T_{i1} \dots T_{in_i}$ for each i, where $x_{s_1} \not\triangleq x_{s_2}$, then take $\vec{V} \triangleq \emptyset$ and $\vec{U}_r$ such that $U_{s_i} \triangleq \lambda \vec{z}_{n_i}.y_i$.
- If each T_i has the form $T_i \triangleq x_s T_{i1} \dots T_{in_i}$, where $n_1 \neq n_2$, then we exploit this difference. Say $n_1 > n_2$, then we may choose $\vec{V} \triangleq V_1 y_2 \dots y_2$ with $n_1 - n_2 + 1$ components in total, where $V_1 \triangleq \lambda \vec{z}_{n_1 - n_2}.y_1$. Then take $\vec{U}_r$ such that $U_s \triangleq \lambda \vec{z}_{n_1+1}.z_{n_1+1}$.

Inductive cases ($i = 1, 2$).

- The easy case is when at least one λ-term T_i begins with a λ-abstraction. For each i, define $T'_i \triangleq U$ if T_i is of the form $T_i \triangleq \lambda z.U$ and $T'_i \triangleq T_i z$ otherwise. In other words, we are applying in both cases the T_i to a variable z which is, for sure, λ-abstracted in the former case. In the latter one, z is obviously free, so we have to cope with it.

  Anyway, T'_i are still $\beta\eta$-normal λ-terms and, moreover, it turns out that $llg(T'_1) + llg(T'_2) < llg(T_1) + llg(T_2)$. So by induction hypothesis there are $\vec{U}_r, U_z, \vec{V}$ with free variables in $\{y_1, y_2\}$ such that $T_i[\vec{x}_r := \vec{U}_r] U_z \vec{V} \to^*_\beta T'_i[\vec{x}_r := \vec{U}_r, z := U_z]\vec{V} \to^*_\beta y_i$.
- In every other case, both λ-terms have not λ-abstractions and do have the same head variable and number of arguments; that is, $T_i \triangleq x_s T_{i1} \dots T_{in}$. We have then $T_{1j} \not\equiv_{\beta\eta} T_{2j}$ and by induction hypothesis there are $\vec{P}_r, \vec{Q} \triangleq Q_1 \dots Q_p$ with free variables in $\{y_1, y_2\}$ such that $T_{ij}[\vec{x}_r := \vec{P}_r]\vec{Q}_p \to_\beta y_i$. It is now the time to use the lemma above, letting $M \triangleq T_{ij}[x_m := P_m]_{1 \le m \le r,\ m \neq s}$, $x \triangleq x_s$, $N \triangleq P_s$ and $y \triangleq y_i$. Then the lemma yields integers $k \geq n+p$, l_i, l'_i such that $T_{ij}[\vec{x}_r := \vec{P'}_r]\vec{Q}_p \mathbf{t}^{l_i} \to^*_\beta y_i \mathbf{t}^{l'_i}$, where $P'_s \triangleq \langle P_s, \lambda \vec{z}_n.z_j \rangle_k$ and $P'_m \triangleq P_m$ if $m \neq s$. We may suppose that, by adding the same constant c_i to l_i, l'_i, $l_1 = l_2 \triangleq l \geq k$. We obtain then $T_i[\vec{x}_r := \vec{P'}_r]\vec{Q}_p \mathbf{t}^{k-n-p}\mathbf{f}\mathbf{t}^{l-k+n+p} \to^*_\beta \langle \dots, T_{ij}[\vec{x}_r := \vec{P'}_r]\vec{Q}_p \mathbf{t}^{k-n-p} \rangle_0 \mathbf{f}\mathbf{t}^{l-k+n+p} \to_\beta T_{ij}[\vec{x}_r := \vec{P'}_r]\vec{Q}_p \mathbf{t}^{l} \to^*_\beta y_1 \mathbf{t}^{l'_i}$. At the end, we need to make some "garbage collection" of the list of $\mathbf{t}$ placed after each y_i. Thus, we can take $\vec{U}_r \triangleq \vec{P'}_r[y_i := \lambda \vec{z}_{l'_i}.y_i]$ and $\vec{V} \triangleq \vec{Q}_p[y_i := \lambda \vec{z}_{l'_i}.y_i]\mathbf{t}^{k-n-p}\mathbf{f}\mathbf{t}^{l-k+n+p}$.

□

At this point, the reader should note the strong similarities between the technique used by Joly and the $\mathbf{P}_n$ utilization. We may state that Joly's proof its a compact coding of the intuition behind the use of n-permutators.

Three are the proof features we may point out. First of all, the role of the lemma 2.14: it is used to have the same result y_i using pairs carrying extraction information. The second point to stress is the role of the natural k. It must be large enough to permit extraction information (projections and external pair selectors) enter the pair, ready to be used at a deeper level of the induction process. Finally, the statement of the theorem is slightly different from the classical one. There are no important differences though; indeed, taking the λ-closure of the two λ-terms to separate and replacing y_1, y_2 with $\mathbf{F}$ and $\mathbf{T}$ respectively, we get back the classical Böhm's theorem. At Appendix A, there is an example of separation using the method just presented.

2.5 Generalizations of Böhm's theorem.

A first generalization of the above theorem considers the case of non-normalizable λ-terms. The problem has been addressed, independently, by Hyland in [8] and Wadsworth in [14] in relation with the meaning of two different λ-terms in Scott's D_∞ model. Indeed, the $\beta\eta$-equivalence is not maximal on such λ-terms and it is necessary to consider the congruence $\mathcal{H}^*$ equating two λ-terms whenever they have the same Böhm tree up to possibly infinite η-expansions. Then, it is possible to prove that for all closed λ-terms $M \not\equiv_{\mathcal{H}^*} N$ there is a sequence of λ-terms $\vec{L}$ such that $M\vec{L}$ β-reduces to the identity $\lambda x.x$ while $N\vec{L}$ is unsolvable [1, §8.3], or vice versa. The property is called of *semi-separation* due to its asymmetry and the fact that it is a weaker result than the one developed by Böhm.

Finally, Böhm et al. [4] extend the technique discussed in this report to the separation of a finite set of arbitrary normal λ-terms.

3 The resource λ-calculus

3.1 Introduction

The resource λ-calculus, denoted $\Lambda^{\mathcal{R}}$, is an extension of the pure λ-calculus allowing to model resource consumption. The first extension in this sense is present in Boudol's λ-calculus with multiplicities [3], where the application is equipped with a resource notion.

More recently, following basic ideas of Girard's quantitative semantics of linear logic, Ehrhard and Regnier have introduced the differential λ-calculus [5], where they not only ported the study of differentiation in λ-calculus but also endowing the set of λ-terms with a structure of vector space, or module, by which is possible to form linear combinations of λ-terms subject to the typical algebra identities. In this way, their study permits to identify the mathematical ground on which can be formally define the property of a λ-term being linear.

Intuitively, a λ-term is said to be linear if it uses its argument exactly once. On the other hand, Girard's linear logic [7], by decomposing intuitionistic implication, makes the thematics of "resource" usage and linearity prominent, relating it with the usual algebraic sense. The work made on differential λ-calculus gets these notions of linearity unified.

The resource λ-calculus studied here is a direct descendant of the differential λ-calculus. In this report, we will work with a slightly different characterization of the calculus specified by Pagani and Tranquilli in [13], but which preserves the same properties. This calculus extends the pure version in two directions.

First of all, $\Lambda^{\mathcal{R}}$ is resource sensitive. It introduces the notion of resource and makes the distinction between linear available resources and infinitely available ones. The second feature of $\Lambda^{\mathcal{R}}$ is the non-determinism. Indeed, in every application the argument is no more a single term but, instead, a bag of resources. At the evaluation time several computational results arise, corresponding to the different choices of distributing the resources among the occurrences of the formal parameter. All possible results are expressed in a finite formal sum of terms.

3.2 Basics

Syntax. The resource λ-calculus has three syntactic categories: resource λ-terms (Λ^r) that are in functional position, bags (Λ^b) that are in argument position and are compounds of resources[6], and finite formal sums that represent the possible results of an evaluation. Formally, the sets Λ^r and Λ^b, and respective sums, are generated by the following grammar:

terms (Λ^r)	$M, N, L ::= x \mid \lambda x.M \mid MP$
bags (Λ^b)	$P, Q, R ::= [M_1, \ldots, M_n, \mathbb{M}^!]$
expressions (Λ^e)	$A, B ::= M \mid P$
sums of terms (Λ^{r+})	$\mathbb{M}, \mathbb{N} ::= M \mid 0 \mid \mathbb{M} + \mathbb{N}$
sums of bags (Λ^{b+})	$\mathbb{P}, \mathbb{Q} ::= P \mid 0 \mid \mathbb{P} + \mathbb{Q}$
sums of expressions (Λ^{e+})	$\mathbb{A}, \mathbb{B} ::= \mathbb{M} \mid \mathbb{P}$

At the syntax level, there are two big differences with the λ-calculus. Obviously, the presence of bags. A bag $[\vec{M}, \mathbb{M}^!]$ is a compound object consisting of a multiset of linear resources $[\vec{M}]$ and a finite sum $\mathbb{M}$ of terms representing reusable resources. This distinction is crucial: while linear resources must be used exactly once during a reduction, the reusable ones can be used as many times as needed[7]. We will deal with bags as multisets presented in multiplicative notation where the union, defined by $[\vec{M}, \mathbb{M}^!] \cdot [\vec{N}, \mathbb{N}^!] \triangleq [\vec{M}, \vec{N}, (\mathbb{M} + \mathbb{N})^!]$, is commutative and associative. Moreover, it has the empty bag $1 \triangleq [0^!]$ as neutral element. To avoid confusion with application we will never omit the dot "$\cdot$". Moreover, to keep things simple, we will write $[L_1, \ldots, L_k]$ for the bag $[L_1, \ldots, L_k, 0^!]$ and $[M^k]$ for the bag $[M, \ldots, M]$ containing k instances of the resource M. With such notation we will be able to manipulate bags as in the following example:

$$[x, y, (x + y)^!] = [x] \cdot [y, (x + y)^!] = [y, x^!] \cdot [x, y^!] = \ldots$$

The second important difference is the presence of sums with the usual properties (commutative, associative and with 0 as neutral element). An important remark here is that the sum we are considering is an idempotent one; that is, for all expressions A we have that $A + A = A$. For this reason, we will say that this kind of resource λ-calculus is *qualitative*.

[6] Resources are no more than resource λ-terms placed inside a bag.
[7] Following Girard's linear logic notation, $\mathbb{M}$ is decorated with a ! superscript.

The *size* of $\mathbb{A} \in \Lambda^{e+}$ is inductively defined as:

$$size([M_1, \ldots, M_k, \mathbb{M}^!]) \triangleq \sum_{i=1}^{k} size(M_i) + size(\mathbb{M}) + 1 \quad size(\lambda x.M) \triangleq \lambda x.size(M) + 1$$
$$size(MP) \triangleq size(M) + size(P) + 1 \qquad size(\sum_i A_i) \triangleq \sum_i size(A_i) \qquad size(x) \triangleq x$$

Speaking about sums, the reader should notice that the grammar allows sums of terms and bags only under the scope of a $(\cdot)^!$ or at the top level. However, we would like to express the typical algebraic operation, thus as a syntactic sugar we extend all the constructors to sums. In fact, as in differential λ-calculus, all the constructors except the $(\cdot)^!$ are (multi)linear[8]; that is, they distribute over sums. Thus, we will permit us the use of the following writings:

$$\lambda x. \sum_{i \in I} M_i \triangleq \sum_{i \in I} \lambda x.M_i \qquad \Big(\sum_{i \in I} M_i\Big)\Big(\sum_{j \in J} P_j\Big) \triangleq \sum_{(i,j) \in I \times J} M_i P_j$$

$$\Big[\Big(\sum_{i \in I} M_i\Big)\Big] \cdot \Big(\sum_{j \in J} P_j\Big) \triangleq \sum_{(i,j) \in I \times J} [M_i] \cdot P_j$$

Rules above are not extended to the operator $(\cdot)^!$ because the idea is that a reusable sum $(M + N)^!$ represents a resource that can be used several times and each time one can choose non-deterministically between M and N.

Thanks to this meta-syntax, we may write something like $(x_1 + x_2)([\lambda y_1.y_1 + y_2] + [(z_1 + z_2)^!])$ instead of $x_1[\lambda y_1.y_1] + x_1[y_2] + x_1[(z_1 + z_2)^!] + x_2[\lambda y_1.y_1] + x_2[y_2] + x_2[(z_1 + z_2)^!]$. Notice here that 0 nullify every term or bag, unless it does not appear under a $(\cdot)^!$.

Rules of substitution are updated to cope with the different kinds of resources. The usual capture-free λ-calculus substitution is written here as $A\{N/x\}$ and is used in the case N is reusable. For linear resources instead, we will define a linear substitution, denoted $A\langle N/x\rangle$, expressing the idea that we must replace the linear resource N to exactly one linear free occurrence of x. In the former case, the substitution process is extended by linearity on sum $\mathbb{A}$. In the latter one, it is extended by bilinearity in both $\mathbb{A}$ and $\mathbb{N}$. Formally, linear substitution is defined as ($y \neq x$ and $y \notin FV(N)$):

$$x\langle N/x\rangle \triangleq N \qquad (\lambda y.M)\langle N/x\rangle \triangleq \lambda y.M\langle N/x\rangle$$
$$y\langle N/x\rangle \triangleq 0 \qquad (MP)\langle N/x\rangle \triangleq (M\langle N/x\rangle)P + M(P\langle N/x\rangle)$$
$$[\mathbb{M}^!]\langle N/x\rangle \triangleq [\mathbb{M}\langle N/x\rangle, \mathbb{M}^!] \qquad ([M] \cdot P)\langle N/x\rangle \triangleq [M\langle N/x\rangle] \cdot P + [M] \cdot P\langle N/x\rangle$$

A remark on linear substitution is necessary when there are multiple choices. In this case, the result will be made of all the possible substitutions and expressed as a formal sum. For example: $(x[x])\langle I/x\rangle = I[x] + x[I]$. Many lemmas and theorems of the pure λ-calculus can be adapted and proved [5] in $\Lambda^{\mathcal{R}}$, as for example the substitution lemma [1, §2.1] stating, under some hypothesis, the possibility to commute the order of a sequence of substitutions (in the current case, the linear ones).

Operational semantic. Always by reduction rules, we will now give the operational semantic of the present calculus which is, not surprisingly, a resource sensitive version of the one defined for the pure λ-calculus.

Definition 3.1. Given a relation $\mathtt{R} \subseteq \Lambda^e \times \Lambda^{e+}$ its *context closure* is the smallest relation in $\Lambda^{e+} \times \Lambda^{e+}$ containing $\mathtt{R}$ and satisfying the following rules:

$$\frac{M \;\mathtt{R}\; \mathbb{M}}{\lambda x.M \;\mathtt{R}\; \lambda x.\mathbb{M}}\,\mathtt{lam} \qquad \frac{M \;\mathtt{R}\; \mathbb{M}}{MP \;\mathtt{R}\; \mathbb{M}P}\,\mathtt{appl} \qquad \frac{P \;\mathtt{R}\; \mathbb{P}}{MP \;\mathtt{R}\; M\mathbb{P}}\,\mathtt{appr}$$

$$\frac{M \;\mathtt{R}\; \mathbb{M}}{[M] \cdot P \;\mathtt{R}\; [\mathbb{M}] \cdot P}\,\mathtt{lin} \qquad \frac{M \;\mathtt{R}\; \mathbb{M}}{[M^!] \cdot P \;\mathtt{R}\; [\mathbb{M}^!] \cdot P}\,\mathtt{bng} \qquad \frac{A \;\mathtt{R}\; \mathbb{A}}{A + \mathbb{B} \;\mathtt{R}\; \mathbb{A} + \mathbb{B}}\,\mathtt{sum}$$

[8]Formally, an operator $F(-)$ (resp. $F(-,-)$) is linear (resp. bilinear) if $F(\sum_i A_i) = \sum_i F(A_i)$ (resp. $F(\sum_i A_i, \sum_j B_j) = \sum_{i,j} F(A_i, B_j)$) holds.

Definition 3.2. A relation $\mathtt{R} \subseteq \Lambda^{e+} \times \Lambda^{e+}$ is *compatible* if it satisfies the following rules:

$$\frac{\mathbb{M}\ \mathtt{R}\ \mathbb{N}}{\lambda x.\mathbb{M}\ \mathtt{R}\ \lambda x.\mathbb{N}}\ \mathtt{lam} \qquad \frac{\mathbb{M}\ \mathtt{R}\ \mathbb{N} \qquad \mathbb{P}\ \mathtt{R}\ \mathbb{Q}}{\mathbb{M}\mathbb{P}\ \mathtt{R}\ \mathbb{N}\mathbb{Q}}\ \mathtt{app} \qquad \frac{\mathbb{M}\ \mathtt{R}\ \mathbb{N}}{[\mathbb{M}^!]\ \mathtt{R}\ [\mathbb{N}^!]}\ \mathtt{bng}$$

$$\frac{\mathbb{M}\ \mathtt{R}\ \mathbb{N} \qquad \mathbb{P}\ \mathtt{R}\ \mathbb{Q}}{[\mathbb{M}] \cdot \mathbb{P}\ \mathtt{R}\ [\mathbb{N}] \cdot \mathbb{Q}}\ \mathtt{lin} \qquad \frac{\mathbb{A}\ \mathtt{R}\ \mathbb{B} \qquad \mathbb{A}'\ \mathtt{R}\ \mathbb{B}'}{\mathbb{A} + \mathbb{A}'\ \mathtt{R}\ \mathbb{B} + \mathbb{B}'}\ \mathtt{sum}$$

As in the pure λ-calculus, the main reduction notion of $\Lambda^{\mathcal{R}}$ is the β-reduction, which is defined as the context closure of the following rule (β-rule):

$$(\lambda x.M)[L_1, \ldots, L_k, \mathbb{N}^!] \to_\beta M\langle L_1/x\rangle \ldots \langle L_k/x\rangle\{\mathbb{N}/x\}$$

An important remark for the β-rule is that the order of linear substitutions is non-influential thanks to the substitution lemma, but it is fundamental to use all the linear resources available. Thus, notice that reducing a term where there is a mismatch between the number of linear resources needed by the functional part of the application and the actual ones it receives, either more or less than expected, leads to an empty sum. For example, terms like $(\lambda x.x)[y, y]$ and $(\lambda x.x)1$ reduce to 0. However, in cases where the number of linear resources needed is greater than the one available, reusable resources can be used as replacement, once all the linear ones are placed. For example, let's consider the term $(\lambda x.xx)[z, y^!]$ which reduces to $zy + yz$.

The pure λ-calculus can be expressed by means of resource λ-calculus by translating every application MN into $M[N^!]$. Indeed, in Λ an actual parameter is copied and used as many times as needed. Hence, we will be able to write ever looping terms like $\mathbf{\Omega} \triangleq (\lambda x.x[x^!])[(\lambda x.x[x^!])^!]$.

It is not surprising that we can express the η-reduction as the context closure of the following rule (η-rule):

$$\lambda x.M[x^!] \to_\eta M, \text{ if } x \notin FV(M)$$

We will omit here all formal definitions concerning normal forms and related stuff because they are really close to the ones given for Λ. The reader should only remember to extend, in a very obvious way, those notions to sums.

3.3 The Taylor expansion

As mentioned before, this calculus is really close to the differential λ-calculus whose key computational ingredient is a notion of derivative, thus capable to express a Taylor expansion of a term. This operation defined in [5] is a translation developing every application as an infinite series of finite applications with rational coefficients. In our resource calculus the Taylor expansion of an expression is a possibly infinite set of finite expressions, corresponding to the support[9] of the operation developed by Ehrhard and Regnier. The terms (resp. bags and expressions) resulting from Taylor expansions are called finite and their set is denoted by Λ^r_{f} (resp. Λ^b_{f} and Λ^e_{f}). It turns out that the above sets are closed under β-reduction. The resulting calculus, denoted $\Lambda^{\mathcal{R}}_{\mathrm{f}}$, is a fragment of $\Lambda^{\mathcal{R}}$ called *finite* resource λ-calculus whose bags contain only linear resources.

Definition 3.3. Let $\mathbb{A} \in \Lambda^{e+}$. The *Taylor expansion* of $\mathbb{A}$ is the set $\mathbb{A}^\circ \subseteq \Lambda^e_{\mathrm{f}}$ which is inductively defined as[10]:

$$x^\circ \triangleq \{x\} \qquad \Big(\sum_i A_i\Big)^\circ \triangleq \bigcup_i A_i^\circ \qquad (\lambda x.M)^\circ \triangleq \lambda x.M^\circ$$

$$(MP)^\circ \triangleq M^\circ P^\circ \qquad ([\mathbb{M}^!])^\circ \triangleq \mathcal{M}_{\mathrm{f}}(\mathbb{M}^\circ) \qquad ([M] \cdot P)^\circ \triangleq [M^\circ] \cdot P^\circ$$

Let us compute, as a simple exercise, the Taylor expansion of the term $(\lambda x.x[x^!])^\circ)$ and note its infinite characterization. Indeed, the result is the infinite set of finite terms $\{\lambda x.x[x^n] \mid n \in \mathcal{N}\}$.

The presence of linear resources makes possible situations where $M^\circ \subsetneq N^\circ$, as for example $M \triangleq x[x, x^!]$, $N \triangleq x[x^!]$ which have, respectively, the following Taylor expansions: $\{x[x^{n+1}] \mid n \in \mathcal{N}\} \subsetneq \{x[x^m] \mid m \in \mathcal{N}\}$.

[9]That is, the set of those finite terms appearing in the series with non-zero coefficient.
[10]Note that the symbol $\mathcal{M}_{\mathrm{f}}(\mathcal{X})$, given a set $\mathcal{X}$, denotes the set of all finite multisets over $\mathcal{X}$.

3.4 Böhm's theorem for resource λ-calculus

We will now present a Böhm-like result for the resource λ-calculus. In [6] is given the intuition that the Taylor expansion of a λ-term could be a resource conscious improvement of the Böhm tree of the same term. Following that intuition, in [11] Manzonetto and Pagani have proved a separation theorem that can be seen as a resource version of the classical one. As the reader will understand at the end of this section, their result is much similar to Hyland's semi-separation.

Technically, the result proved by the two is quite different from the one proved by Böhm. Indeed, we saw before that a crucial ingredient in that proof is the possibility to select and erase subterms in order to pull out their structural difference. This is not directly possible in $\Lambda^{\mathcal{R}}$ because linear resources must be consumed and cannot be erased. Moreover, the separation process considers a different condition to work on because it turns out that there are distinct resource λ-terms in $\mathrm{NF}_{\beta\eta}$ which are inseparable. The final result is based instead on the Taylor expansion of terms since all normal forms having equal Taylor expansions are inseparable, as proved by Manzonetto in [10].

Express the Taylor expansion syntactically. To achieve an internal separation based on the Taylor expansion of terms, we need to syntactically characterize it. Indeed, because of its infinite nature, the property of having the same Taylor expansion is more semantical than syntactical. Thus, we introduce the following congruence stressing the relation between a reusable resource and its first approximations applying the Taylor expansion on it. It is really what we expect because, roughly speaking, is telling us that a reusable resource can be thought as a sum of a zeroary linear instance (i.e. 1), a linear instance and still its reusable part.

Definition 3.4. The *Taylor equivalence* $\equiv_\tau$ is the congruence generated by:

$$[M^!] \equiv_\tau 1 + [M, M^!]$$

Moreover, we set $\mathbb{A} \sqsubseteq_\tau \mathbb{B}$ iff $\mathbb{A} + \mathbb{B} \equiv_\tau \mathbb{B}$.

It is not difficult to see that $\sqsubseteq_\tau$ is a preorder since $\equiv_\tau$ is a congruence. Moreover, $\sqsubseteq_\tau$ is compatible. Let us stress only the case for the `bng` rule since the other cases follow by (bi)linearity. We have to prove that $[\mathbb{A}^!] \sqsubseteq_\tau [\mathbb{B}^!]$, that is $[\mathbb{A}^!] + [\mathbb{B}^!] \equiv_\tau [\mathbb{B}^!]$. From the hypothesis $\mathbb{A} + \mathbb{B} \equiv_\tau \mathbb{B}$ we obtain $[(\mathbb{A} + \mathbb{B})^!] \equiv_\tau [\mathbb{B}^!]$, and so $[\mathbb{A}^!] \cdot [\mathbb{B}^!] \equiv_\tau [\mathbb{B}^!]$. Hence, $[\mathbb{A}^!] + [\mathbb{B}^!] \equiv_\tau [\mathbb{A}^!] + [\mathbb{A}^!] \cdot [\mathbb{B}^!] \equiv_\tau [\mathbb{A}^!](1+[\mathbb{B}^!]) \equiv_\tau [\mathbb{A}^!](1 + 1+[\mathbb{B}, \mathbb{B}^!]) \equiv_\tau [\mathbb{A}^!](1+[\mathbb{B}, \mathbb{B}^!]) \equiv_\tau [\mathbb{A}^!] \cdot [\mathbb{B}^!] \equiv_\tau [\mathbb{B}^!]$.

The first step toward the proof that $\sqsubseteq_\tau$ captures the inclusion between Taylor expansions is the study of the linear underlying structure of finite terms.

Definition 3.5. Given $A \in \Lambda^e$, its *skeleton* $\mathbb{s}(A) \in \Lambda^e_{\mathrm{f}}$ is obtained by erasing all the reusable resources occurring in A. That is, inductively:

$$\mathbb{s}(x) \triangleq x \qquad \mathbb{s}(\lambda x.M) \triangleq \lambda x.\mathbb{s}(M)$$
$$\mathbb{s}(MP) \triangleq \mathbb{s}(M)\mathbb{s}(P) \qquad \mathbb{s}([M_1, \dots, M_n, \mathbb{M}^!]) \triangleq [\mathbb{s}(M_1), \dots, \mathbb{s}(M_n)]$$

One would like to have: $\mathbb{s}(A) \in B^\circ$ entails $A \sqsubseteq_\tau B$. But in general, this is not the case. Take for instance $A \triangleq x[x^!]$ and $B \triangleq x[y^!]$; we have $\mathbb{s}(A) = x1 \in \{x[y^n] \mid n \in \mathcal{N}\} = B^\circ$ but $A \not\sqsubseteq_\tau B$. The problem here is that they do have different resource structures (i.e. x and y), but the skeleton operation on A elude them. The idea is then to require more information inside bags, a larger multiset of linear resources of the same kind of the reusable ones essential to a finer comparison.

Definition 3.6. Given $k \in \mathcal{N}$, we say that $A \in \Lambda^e$ is *k-expanded* if, whenever it contains a bag that can be decomposed into $[M^!] \cdot P$, we have $P = [M^k] \cdot P'$, for some P' k-expanded. A sum $\mathbb{A} \in \Lambda^{e+}$ is k-expanded if all its summands are.

With the k-expanded condition, it turns out that the above desired implication is now provable.

Lemma 3.7. *Let $A \in \Lambda^e$ be a k-expanded for some $k \in \mathcal{N}$. Then for every $B \in \Lambda^e$ such that size$(B) \leq k$, we have that $\mathbb{s}(A) \in B^\circ$ entails $A \sqsubseteq_\tau B$.*

We will not prove this lemma [11]. Intuitively, the side condition on the size of B is needed to imply the presence of a reusable part and an amount of linear resources less or equal to the ones in A. Indeed, take another look at the last example in 3.3.

The k-expanded condition is not too constraining. In fact, it is always possible to extract from the reusable part the needed amount of linear resources, obtaining eventually a k-expanded sum.

Lemma 3.8. *For all $A \in \Lambda^e$ and $k \in \mathcal{N}$, there is a k-expanded sum $\mathbb{A}$ such that $A \equiv_\tau \mathbb{A}$.*

The reader can find the easy proof in [11]; here, we will give an example. The bag $[x, x^!]$ is 1-expanded, and we want it to be 3-expanded. Exploiting the Taylor equivalence and some algebraic operations we have: $[x, x^!] = [x] \cdot [x^!] \equiv_\tau [x] \cdot (1 + [x, x^!]) = [x] + [x^2] \cdot [x^!] \equiv_\tau [x] + [x^2] \cdot (1 + [x, x^!]) = [x] + [x^2] + [x^3, x^!]$. Note that all the summands are 3-expanded and $[x, x^!] \equiv_\tau [x] + [x^2] + [x^3, x^!]$.

Proposition 3.9. *For all $\mathbb{A}, \mathbb{B} \in \Lambda^{e+}$ we have that $\mathbb{A} \sqsubseteq_\tau \mathbb{B}$ iff $\mathbb{A}^\circ \subseteq \mathbb{B}^\circ$.*

Proof. ($\Rightarrow$) By induction on the derivation of $\mathbb{A} + \mathbb{B} \equiv_\tau \mathbb{B}$, remarking that all rules defining $\equiv_\tau$ preserve the property of having the same Taylor expansion.
($\Leftarrow$) By induction on the number of summands in $\mathbb{A}$. If $\mathbb{A} = 0$, then we have $0 + \mathbb{B} \equiv_\tau \mathbb{B}$, and so by reflexivity of $\equiv_\tau$ we can conclude $\mathbb{A} \sqsubseteq_\tau \mathbb{B}$. If $\mathbb{A} = A + \mathbb{A}'$, then by induction hypothesis we obtain $\mathbb{A}' \sqsubseteq_\tau \mathbb{B}$. Let $k \geq max(size(A), size(\mathbb{B}))$, by lemma 3.8, there is a k-expanded sum $\mathbb{A}'' = A_1 + \cdots + A_n \equiv_\tau A$. From $\mathbb{A}'' \sqsubseteq_\tau A \Rightarrow (\mathbb{A}'' \sqsubseteq_\tau A) \wedge (A \sqsubseteq_\tau \mathbb{A}'')$ and the already proved ($\Rightarrow$) direction of the proposition we know that $(\mathbb{A}'')^\circ = A^\circ \subseteq \mathbb{B}^\circ$. This means that, $\forall i \leq n\ (A_i)^\circ \subseteq \mathbb{B}^\circ$. In particular $\mathbb{s}(A_i) \in \mathbb{B}^\circ_{j_i}$, for some particular summand of $\mathbb{B}$. Since $\mathbb{A}''$ is k-expanded entails that A_i are too. Moreover, $size(B_{j_i}) \leq size(\mathbb{B}) \leq k$, applying lemma 3.7 we get $A_i \sqsubseteq_\tau B_{j_i} \sqsubseteq_\tau \mathbb{B}$. Since this hold for every $i \leq n$, we obtain $\mathbb{A}'' \sqsubseteq_\tau \mathbb{B}$. Then we can conclude $\mathbb{A} \equiv_\tau \mathbb{A}' + \mathbb{A}'' \sqsubseteq_\tau \mathbb{B} + \mathbb{B} \equiv_\tau \mathbb{B}$. ☐

Separate terms in resource λ-calculus. As stated before, given $\mathbb{M}, \mathbb{N} \in \Lambda^{r+}$ such that $\mathbb{M} \equiv_\tau \mathbb{N}$, they are inseparable even if distinct $\beta\eta$-normal forms. Thus, as major difference from the pure calculus case, to achieve an internal separation we will investigate on β-normal $\tau\eta$ distinct terms.

A first idea one could think about separating two resource terms $\mathbb{M} \not\equiv_{\tau\eta} \mathbb{N}$ could be to study their η-normal forms and checking whenever they are τ-different. The problem is that this does not hold in general, namely there are τ-different η-normal forms $\mathbb{N}, \mathbb{N}'$ coming from terms $\mathbb{M}, \mathbb{M}'$ such that $\mathbb{M} \equiv_\tau \mathbb{M}'$. For all variable x, y, consider $\mathbb{M}_{x,y} \triangleq v[\lambda z.x1, \lambda z.y[z^!]] + v[\lambda z.x[z, z^!], \lambda z.y[z^!]]$, $\mathbb{M}_{y,x}$ as the previous but with variables x, y exchanged, and compute their Taylor expansion:

$$\begin{aligned}(\mathbb{M}_{x,y})^\circ &\triangleq \{v[\lambda z.x, \lambda z.y[z^n] \mid n \in \mathcal{N}\} \cup \{v[\lambda z.x[z^{u+1}], \lambda z.y[z^v]] \mid u, v \in \mathcal{N}\} \\ &= \{v[\lambda z.x[z^m], \lambda z.y[z^n]] \mid m, n \in \mathcal{N}\}\end{aligned}$$

Clearly, the term $\mathbb{M}_{y,x}$ has the same Taylor expansion. Hence, by proposition 3.9, the fact that $(\mathbb{M}_{x,y})^\circ = (\mathbb{M}_{y,x})^\circ$ entails $\mathbb{M}_{x,y} \equiv_\tau \mathbb{M}_{y,x}$. Consider now their respective η-normal forms:

- $\mathbb{N}_{x,y} \triangleq v[\lambda z.x1, y] + v[\lambda z.x[z, z^!], y]$
- $\mathbb{N}_{y,x} \triangleq v[\lambda z.y1, x] + v[\lambda z.y[z, z^!], x]$

and compute the Taylor expansion of both. The result is the following:

- $(\mathbb{N}_{x,y})^\circ = \{[\lambda z.x[z^n], y] \mid n \in \mathcal{N}\}$
- $(\mathbb{N}_{y,x})^\circ = \{[\lambda z.y[z^n], x] \mid n \in \mathcal{N}\}$

Thus, by proposition 3.9, $(\mathbb{N}_{x,y})^\circ \neq (\mathbb{N}_{y,x})^\circ$ entails $\mathbb{N}_{x,y} \not\equiv_\tau \mathbb{N}_{y,x}$. That is, the Taylor equivalence $\equiv_\tau$ is not preserved by η-reductions.

To overcome these problems, the idea developed in [11] is, intuitively, to compute linear approximations of the two terms $\mathbb{M}, \mathbb{N}$ using the Taylor expansion and to analyze their finite η-normal forms pointwise. In other words, we will compute the Taylor expansion of the terms and investigate if both are reducible to the other by finite η-reductions; this would mean that they are $\tau\eta$-indistinguishable therefore not separable. Following this intuition, we will introduce a η-reduction on $\Lambda^{\mathcal{R}}_{\mathrm{f}}$ and a relation $\preceq_s$ such that $\mathbb{M} \not\equiv_{\tau\eta} \mathbb{N}$ entails either $\mathbb{M} \not\preceq_s \mathbb{N}$ or vice versa.

Definition 3.10. The *partial η-reduction* $\rightarrow_\varphi$ is the contextual closure of the rule defined on $\Lambda^{\mathcal{R}}_{\mathrm{f}}$: $\lambda x.M[x^n] \rightarrow_\varphi M$ if $x \notin FV(M)$.

Definition 3.11. Given $\mathbb{M}, \mathbb{N} \in \Lambda^{r+}$, we define:

- $\mathbb{M} \preceq_s \mathbb{N}$ iff $\forall M \in \mathbb{M}^\circ, \exists N \in \mathbb{N}^\circ$ such that $M \rightarrow^*_\varphi N$
- $\mathbb{M} \preceq_\tau \mathbb{N}$ iff $\exists \mathbb{M}' \rightarrow^*_\eta \mathbb{M}, \exists \mathbb{N}' \rightarrow^*_\eta \mathbb{N}$ such that $\mathbb{M}' \sqsubseteq_\tau \mathbb{N}'$

Remark 3.12. Note that $\mathbb{M} \to_\eta \mathbb{N}$ implies $\mathbb{M} \preceq_s \mathbb{N}$. In fact, taking into account the case of only one summand, if $\lambda x.N[x^!] \to_\eta N$ then obviously $\lambda x.N^\circ[x^n] \to_\varphi^* N^\circ$. Moreover, since it is not difficult to check that $\preceq_s$ is a preorder, by transitivity of $\preceq_s$ we have that $\mathbb{M} \to_\eta^* \mathbb{N}$ entails $\mathbb{M} \preceq_s \mathbb{N}$.

Lemma 3.13. *Let* $\mathbb{M}, \mathbb{N} \in \Lambda^{r+}$. *Then* $\mathbb{M} \preceq_\tau \mathbb{N}$ *and* $\mathbb{N} \preceq_\tau \mathbb{M}$ *entails* $\mathbb{M} \equiv_{\tau\eta} \mathbb{N}$.

Proof. From $\mathbb{M} \preceq_\tau \mathbb{N}$ we have by definition $\mathbb{M}' \to_\eta^* \mathbb{M}$ and $\mathbb{N}' \to_\eta^* \mathbb{N}$ such that $\mathbb{M}' \sqsubseteq_\tau \mathbb{N}'$. Moreover, from $\mathbb{N} \preceq_\tau \mathbb{M}$ we have $\mathbb{N}'' \to_\eta^* \mathbb{N}$ and $\mathbb{M}'' \to_\eta^* \mathbb{M}$ such that $\mathbb{N}'' \sqsubseteq_\tau \mathbb{M}''$. Then, by definition of $\sqsubseteq_\tau$, $\mathbb{N}' \equiv_\tau \mathbb{M}' + \mathbb{N}'$ and applying η-reductions on both sides, $\mathbb{N}' \to_\eta^* \mathbb{N} \equiv_{\tau\eta} \mathbb{M}' + \mathbb{N}' \to_\eta^* \mathbb{M} + \mathbb{N}$, we obtain $\mathbb{N} \equiv_{\tau\eta} \mathbb{M} + \mathbb{N}$. Symmetrically, $\mathbb{M}'' \equiv_\tau \mathbb{M}'' + \mathbb{N}''$ and applying η-reductions on both sides, $\mathbb{M}'' \to_\eta^* \mathbb{M} \equiv_{\tau\eta} \mathbb{M}'' + \mathbb{N}'' \to_\eta^* \mathbb{M} + \mathbb{N}$, we obtain $\mathbb{M} \equiv_{\tau\eta} \mathbb{M} + \mathbb{N}$. Now, we can conclude $\mathbb{M} \equiv_{\tau\eta} \mathbb{N}$. □

The next lemma, which we are not going to prove[11], is the final link of the chain of implications relating $\equiv_{\tau\eta}$ and $\preceq_s$.

Lemma 3.14. *Let* $\mathbb{M}, \mathbb{N} \in \Lambda^{r+}$. *Then* $\mathbb{M} \preceq_s \mathbb{N}$ *entails* $\mathbb{M} \preceq_\tau \mathbb{N}$.

Summarizing, if $\mathbb{M} \not\equiv_{\tau\eta} \mathbb{N}$ then, say, $\mathbb{M} \not\preceq_s \mathbb{N}$. This means that $\exists M \in \mathbb{M}^\circ$ such that $\forall N \in \mathbb{N}^\circ$ we surely have that $M \not\to_\varphi^* N$. We will now prove a lemma who separates such finite term M from all $\mathbb{N}$'s having finite approximations really different from M; that is, such that $\forall N \in \mathbb{N}^\circ$ we have $M \not\to_\varphi^* N$. The reader should notice here that the number of such $\mathbb{N}$ is infinite.

For technical reasons, we will need to suppose that M has a number of λ-abstractions greater than, or equal to, the ones of $\mathbb{N}$ and every subterms of it.

Definition 3.15. Let $M, N \in \Lambda^r$ be in NF_β and of the shape $M \triangleq \lambda\vec{x}_a.y\vec{P}_p$, $N \triangleq \lambda\vec{x}_b.z\vec{Q}_q$. We will say that M is λ-*wider* than N if $a \geq b, p \geq q$ and each (linear or reusable) resource $L \in P_i$ is λ-wider than every (linear or reusable) resource $L' \in Q_i$, for all $i \leq q$. Given $\mathbb{M}, \mathbb{N} \in \mathrm{NF}_\beta$, we will say that $\mathbb{M}$ is λ-wider than $\mathbb{N}$ if each summands of $\mathbb{M}$ is λ-wider than all summands of $\mathbb{N}$.

As an example, the term $x[\lambda y.y, x]$ is λ-wider than $x[y, x]$, but not than itself. Indeed, x is not λ-wider than $\lambda y.y$.

Remark 3.16. Remembering the fact that Taylor expansion keeps by definition λ-abstractions of terms, if $\mathbb{M}$ is λ-wider than $\mathbb{N}$ then every $M \in \mathbb{M}^\circ$ is λ-wider than $\mathbb{N}$.

Lemma 3.17. *For all* $\mathbb{M}, \mathbb{N} \in NF_\beta$, *there is a β-normal* $\mathbb{M}'$ *such that* $\mathbb{M}' \to_\eta^* \mathbb{M}$ *and* $\mathbb{M}'$ *is λ-wider than both* $\mathbb{M}$ *and* $\mathbb{N}$.

We will not detail the proof which is quite straightforward: once computed the maximum between the size of $\mathbb{M}$ and $\mathbb{N}$, we have only to increase accordingly the number of λ-abstractions, $\lambda\vec{x}_i$, to those already present while adding reusable bags of the shape $[\vec{x_i^!}]$. Obviously, we have to apply recursively this process to all linear and reusable resources of $\mathbb{M}$. Note that this does not invalidate our global approach to the problem because if $M \to_\eta^* N$ then it still holds in the case $\lambda\vec{x}_i.M[\vec{x_i^!}]$ whenever we choose variables $x_i \notin FV(M)$, or vice versa.

The next lemma will be the key element for proving the resource Böhm's theorem. The basic idea is simple: a term $M \in \Lambda_{\mathrm{f}}^{\mathcal{R}}$ φ-reduces either to itself if it is in φ-normal form (i.e. we cannot apply the φ-rule anymore) or to a term N such that $M \triangleq \lambda x.N[x^n] \to_\varphi N$. The next lemma exploits this simple observation to separate M from those terms which are not reachable from M by φ-reductions.

We will indicate with $\mathbf{I}$ the λ-term $\lambda x.x$, the identity function. Moreover, we will denote $\mathbf{rP}_n$ the resource λ-term $\lambda\vec{x}_n\lambda x.x[x_1^!]\ldots[x_n^!]$, the resource n-permutator.

Lemma 3.18. *Let* $M \in \Lambda_f^{\mathcal{R}}$ *be a finite β-normal form and* $\Gamma \triangleq \{x_1, \ldots, x_d\}$. *Then, there exist a substitution σ and a sequence $\vec{R}$ of closed bags such that, for all β-normal* $\mathbb{N} \in \Lambda^{r+}$ *such that M is λ-wider than* $\mathbb{N}$ *and* $FV(\mathbb{N}) \subseteq \Gamma$, *we have:*

$$\mathbb{N}\sigma\vec{R} \to_\beta^* \begin{cases} \mathbf{I} & \textit{if } \exists N' \in \mathbb{N}^\circ,\ M \to_\varphi^* N' \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

[11] The reader can find the proof in [11].

Proof. The proof reasons by induction on $size(M)$. To be precise, what we are going to prove is a stronger statement than the one of the lemma, but which implies it. Indeed, we will use an induction loading, namely the fact that we quantify over all the substitutions σ, and σ of the shape $\sigma \triangleq \{\mathbf{rP}_{k_1}/x_1, \ldots, \mathbf{rP}_{k_d}/x_d\}$, where for all distinct $i, j \leq d$ we have k_i, $|k_i - k_j| > 2k$ for some fixed $k > size(M)$. Throughout the entire proof, we will consider the case of $\mathbb{N}$ be a single term N since, as we will explain at the end of the proof, the general case follows by distributing $\sigma\vec{R}$ on every summand of $\mathbb{N}$.

Base case. The term M is a variable, say $M \triangleq x$, and $\Gamma \triangleq \{x\}$. In this case, the only possibility to have (1) reducing to $\mathbf{I}$ comes out when either N is the variable x. In fact for all $\sigma \triangleq \{\mathbf{rP}_{k_x}/x\}$ such that $k_x > 2k$, and by setting $m \triangleq max\{k_x\} \triangleq k_x$, we firstly define a closed term $H \triangleq \lambda w_1 \ldots \lambda_{m+k_x}.\mathbf{I}$ and, finally, we build $\vec{R} \triangleq 1^{k_x}[H^!]1^m$. It is easy to check that $N\sigma\vec{R} \triangleq x\sigma\vec{R} \to_\beta^* \mathbf{I}$, since $x \to_\varphi^* x$ indeed. In all other cases where N is not x, the result will be obviously 0.

Inductive case. In the general case, remember that $\Gamma \triangleq \{x_1, \ldots, x_d\}$. Let term M be

$$M \triangleq \lambda x_{d+1} \ldots \lambda_{d+a}.x_h P_1 \ldots P_p$$

where $a, p \geq 1$, $h \leq d + a$ and, for every $i \leq p$, bag $P_i \triangleq [M_{i,1}, \ldots, M_{i,t_i}]$ with $t_i \geq 0$. Notice that, for every $j \leq t_i$, $FV(M_{i,j}) \subseteq \Gamma \cup \{x_{d+1}, \ldots, x_{d+a}\}$ and $k > size(M) > size(M_{i,j})$. Thus, we can define $\sigma' \triangleq \sigma \cdot \{\mathbf{rP}_{k_{d+1}}/x_{d+1}, \ldots, \mathbf{rP}_{k_{d+a}}/x_{d+a}\}$ assuring k_i, $|k_i - k_j| \geq 2k$ for every distinct $i, j \leq d+a$. Then we can apply then the induction hypothesis on every $M_{i,j}$, obtaining a sequence $\vec{R_{i,j}}$ of closed bags satisfying condition (1) for every N' such that $M_{i,j}$ is λ-wider than N' and $FV(N') \subseteq \Gamma \cup \{x_{d+1}, \ldots, x_{d+a}\}$. We can now build the sequence of closed bags $\vec{R}$ starting from such $\vec{R_{i,j}}$'s. As in the base case, we first define the closed term H as follows (setting $m \triangleq max\{k_1, \ldots, k_{d+a}\}$):

$$H \triangleq \lambda z_1 \ldots \lambda z_p \lambda w_1 \ldots \lambda w_{m+k_h-p}.\mathbf{I}[z_1\vec{R_{1,1}}] \ldots [z_1\vec{R_{1,t_1}}] \ldots [z_p\vec{R_{p,1}}] \ldots [z_p\vec{R_{p,t_p}}]$$

Thus, we can build $\vec{R} \triangleq [\mathbf{rP}^!_{k_{d+1}}] \ldots [\mathbf{rP}^!_{k_{d+a}}]1^{k_h-p}[H^!]1^m$. Notice that when every $i \leq p$, $t_i = 0$; that is, when every $P_i \triangleq 1$ we do not apply the induction hypothesis and the term H will be of the form $\vec{\lambda z_p}\vec{\lambda w}.\mathbf{I}$.

Let the term N be

$$N \triangleq \lambda x_{d+1} \ldots \lambda x_{d+b}.x_{h'} Q_1 \ldots Q_q$$

and let us prove that $N\sigma\vec{R} \to_\beta^* \mathbf{I}$ if there is a $N' \in N^\circ$, $M \to_\varphi^* N'$, otherwise $N\sigma\vec{R} \to_\beta^* 0$. Since M is λ-wider than N, we have $a \geq b$ and

$$N\sigma\vec{R} \to_\beta^* \mathbf{rP}_{k_{h'}} Q_1 \ldots Q_q \sigma' [\mathbf{rP}^!_{k_{d+b+1}}] \ldots [\mathbf{rP}^!_{k_{d+a}}]1^{k_h-p}[H^!]1^m \tag{2}$$

Moreover, since $k > size(M) > a$ and $k > size(M) > q$, we have that $k_{h'} > 2k > q + (a - b)$ and so, setting $g = q + (a - b)$, we have (2) β-reducing to:

$$(\lambda y_{g+1} \ldots \lambda y_{k_{h'}} \lambda y.y Q_1 \ldots Q_q [\mathbf{rP}^!_{k_{d+b+1}}] \ldots [\mathbf{rP}^!_{k_{d+a}}][y^!_{g+1}] \ldots [y^!_{k_{h'}}])\sigma' 1^{k_h-p}[H^!]1^m \tag{3}$$

Let us consider three possible cases keeping in mind the syntactical structure of N.

1. $(h \neq h')$. This is the case where M and N differ on their head variable, hence we have that for all $N' \in N^\circ$, $M \not\to_\varphi^* N'$. Then, we prove (3) β-reducing to 0. By the hypothesis on $k_h, k_{h'}$, we have either $k_h > k_{h'} + 2k$ or $k_{h'} > k_h + 2k$. In the first case, since $k > size(M)$ and M λ-wider than N, we get $k_h > k_{h'} + p + q + a + b > k_{h'} + p - g$ implying $k_h - p > k_{h'} - g$. This means that in (3) the bunch of λ-abstraction are strictly less than the number of the first sequence of 1^{k_h-p}. Thus (3)$\to_\beta^*$ 0 since the head variable y will get 0 from an empty bag. In the second case, since $m \geq k_h, k_{h'}$ we have $m \geq k_{h'} - g > k_h - p$ entailing (3)$\to_\beta^*$ 0 since the head variable y will get 0 from an empty bag of the bunch of the 1^m.

2. $(h = h'$, but $p - a \neq q - b)$. This means that there is a mismatch between the bunch of λ-abstractions and bags in M from those in N. Moreover, notice that the Taylor expansion erases neither λ-abstractions nor bags and $\to_\varphi$ preserves the difference between the two. Then, we prove (3) β-reducing to 0. As before, we have two possible scenario. If $p - a < q - b$, then $k_{h'} - g = k_h - g < k_h - p$ implying (3)$\to_\beta^*$ 0 since the head variable y will get 0 from an empty bag of the bunch of the 1^{k_h-p}. Otherwise, $p - a > q - b$, then $k_{h'} - g = k_h - g > k_h - p$ implying (3)$\to_\beta^*$ 0 since the head variable y will get 0 from an empty bag of the bunch of the 1^m.

3. ($h = h'$ and $p - a = q - b$). In this case we have $k_{h'} - g = k_h - g = k_h - p$ entailing (3) β-reducing to $HQ_1 \dots Q_q[\mathbf{rP}^!_{k_{d+b+1}}] \dots [\mathbf{rP}^!_{k_{d+a}}]1^{k_h-p+m}\sigma'$. By definition of the substitution σ', we can rewrite this term as

$$HQ_1 \dots Q_q[x^!_{k_{d+b+1}}] \dots [x^!_{k_{d+a}}]1^{k_h-p+m}\sigma' \tag{4}$$

Thus substituting to H its definition we get

$$(\lambda\vec{z}_p\lambda\vec{w}_{k_h-p+m}.\mathbf{I}[z_1\vec{R_{1,1}}] \dots [z_1\vec{R_{1,t_1}}] \dots [z_p\vec{R_{p,1}}] \dots [z_p\vec{R_{p,t_p}}])Q_1 \dots Q_q[x^!_{k_{d+b+1}}] \dots [x^!_{k_{d+a}}]1^{k_h-p+m}\sigma' \tag{5}$$

Intuitively, since $p - a = q - b$ entails $p = q + a - b$, the bunch of λ-abstractions $\lambda\vec{z}_p$ will consume the arguments $Q_1 \dots Q_q[x^!_{k_{d+b+1}}] \dots [x^!_{k_{d+a}}]$. Then, the second bunch of λ-abstractions $\lambda\vec{w}_{k_h-p+m}$ will consume all the remaining empty bags 1^{k_h-p+m}. Eventually, (5) will β-reduce to something like

$$\mathbf{I}[?\vec{R_{1,1}}] \dots [?\vec{R_{1,t_1}}] \dots [?\vec{R_{p,1}}] \dots [?\vec{R_{p,t_p}}]\sigma' \tag{6}$$

Obviously, the final result depends on how the terms $?\sigma'\vec{R_{i,j}}$ are going to interact. It will be $\mathbf{I}$ in the case where all $?\sigma'\vec{R_{i,j}} \to^*_\beta \mathbf{I}$, meaning that for all $M_{i,j}$ there exists $?' \in ?^\circ$ such that $M_{i,j} \to^*_\varphi ?'$ implying, eventually, that there exists $N' \in N^\circ$, $M \to^*_\varphi N'$. Otherwise, the final result will be 0.

Let us look at the details. By hypothesis, N could have bags containing reusable resources. Hence, for every $i \le q$, let us explicit $Q_i \triangleq [N_{i,1}, \dots, N_{i,l_i}, (N_{i,l_i+1} + \dots + N_{i,v_i})^!]$, with $v_i \ge l_i \ge 0$. Depending on the availability and type of resources in bags, we split into three subcases. Recall that t_i is the number of linear resources in P_i (since, $M \in \Lambda^{\mathcal{R}}_{\mathrm{f}}$).

(a) ($\exists i \le q,\ t_i < l_i$). In this subcase (5)$\to^*_\beta$ 0. In fact, any $N' \in N^\circ$ is of the form, with $Q'_j \in Q^\circ_j$ for every $j \le q$, $\lambda x_{d+1} \dots \lambda x_{d+b}.x_{h'}Q'_1 \dots Q'_q$. In particular, there is Q'_i with at least $l_i > t_i$ linear resouces entailing the fact that $P_i \not\to^*_\varphi Q'_i$, thus $M \not\to^*_\varphi N'$. Hence, applying the β-reduction to (5) will make eventually match the λ-abstraction λz_i in H with the bag Q_i. The variable z_i has $t_i < l_i$ linear occurrences, leading the final result to 0 due to a mismatch between the linear resources available from those needed.

(b) ($\exists i \le q,\ t_i \ne l_i$ and $v_i = l_i$). This is the case where there is Q_i with no reusable resources and an amount of linear resources different from those present in P_i. Hence, $M \not\to^*_\varphi N'$ for every $N' \in N^\circ$. Obviously, we get (5)$\to^*_\beta$ 0.

(c) ($\forall i \le q,\ t_i \ge l_i$, and $t_i > l_i$ entails $v_i > l_i$). Since Q_i could have reusable resources, we have to keep track of all the possibilities to assign them once used all the linear ones exactly once. Let $\mathcal{F}_i$ be the set of maps $s : \{1, \dots, t_i\} \to \{1, \dots, l_i, l_i + 1, \dots, v_i\}$ such that

- l_i-injectivity: for every $j, h \le t_i$, if $s(j) = s(h)$ then $j = h$
- l_i-surjectivity: for every $h \le l_i$, there is $j \le t_i$, $s(j) = h$

Intuitively, $\mathcal{F}_i$ describes the possible ways of replacing the t_i linear occurrences of the variable z_i in H by the v_i resources of Q_i: the two conditions together assure that each linear resource of Q_i must replace exactly one occurrence of z_i. Remembering the fact that in $\Lambda^{\mathcal{R}}$ this process result in a formal sum and $p - q = a - b$, we get that (5) β-reduces to the sum

$$\sum_{s_1 \in F_1 \dots s_q \in F_q} \mathbf{I}[N_{1,s_1(1)}\vec{R_{1,1}}] \dots [N_{1,s_1(t_1)}\vec{R_{1,t_1}}] \dots [N_{q,s_q(1)}\vec{R_{q,1}}] \dots [N_{q,s_q(t_q)}\vec{R_{q,t_q}}] \tag{7}$$

$$[x_{d+b+1}\vec{R_{q+1,1}}] \dots [x_{d+b+1}\vec{R_{q+1,t_1}}] \dots [x_{d+a}\vec{R_{p,1}}] \dots [x_{d+a}\vec{R_{p,t_p}}]\sigma'$$

Since $p - q = a - b$ could be greater then 0, all the remaining z_i are replaced using reusable resources of bags $[x^!_i]$, with $d + b + 1 \le i \le d + a$.

Notice that every term $M_{i,j}$ is λ-wider than $N_{i,s_i(j)}$ and $FV(N_{i,s_i(j)} \subseteq \Gamma \cup \{x_{d+1}, \dots, x_{d+a}\}$, for all $i \le q$, $s_i \in \mathcal{F}_i$, $j \le t_i$. Thus, by induction hypothesis we obtain

$$N_{i,s_i(j)}\sigma'\vec{R_{i,j}} \to^*_\beta \begin{cases} \mathbf{I} & \text{if } \exists N' \in N^\circ_{i,s_i(j)},\ M_{i,j} \to^*_\varphi N' \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The same holds for every $x_{d+b+i} \in \Gamma \cup \{x_{d+1}, \ldots, x_{d+a}\}$, $1 \leq i \leq p-q = a-b$, $j \leq t_{q+i}$, so by induction hypothesis we get

$$x_{d+b+i}\sigma' \vec{R}_{q+i,j} \to_\beta^* \begin{cases} \mathbf{I} & \text{if , } M_{i,j} \to_\varphi^* x_{d+b+i} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Thanks to the conditions on every function s_i, from (8) and (9) we deduce that (5)$\to_\beta^*$ $\mathbf{I}$ if and only if:

- for all $i \leq q$, $\exists Q_i' \in Q_i^\circ$ such that $P_i \to_\varphi^* Q_i'$, and
- for all $i < p-q = a-b$, $P_{q+i} \to_\varphi^* [x_{d+b+i}^{t_{q+i}}]$

This means that (5)$\to_\beta^*$ $\mathbf{I}$ if and only if

$$\begin{aligned}
&M \triangleq \lambda x_{d+1} \ldots \lambda x_{d+a}.x_h P_1 \ldots P_p \to_\varphi^* \\
&\to_\varphi^* \lambda x_{d+1} \ldots \lambda x_{d+b} \lambda x_{d+b+1} \ldots \lambda x_{d+a}.x_h Q_1' \ldots Q_q' [x_{d+b+1}^{t_{q+1}}] \ldots [x_{d+a}^{t_p}] \to_\varphi^* \\
&\to_\varphi^* \lambda x_{d+1} \ldots \lambda x_{d+b}.x_h Q_1' \ldots Q_q'
\end{aligned}$$

where the last term is in N°.

We have just proved that $N\sigma\vec{R}$ β-reduces to $\mathbf{I}$ if $\exists N' \in N^\circ$ such that $M \to_\varphi^* N'$ and 0 otherwise. Hence, we can conclude saying that condition (1) holds at least in the case where $\mathbb{N}$ is a single term N. At this point, the reader should had understood that distributing $\sigma\vec{R}$ on every summand of $\mathbb{N}$ is sufficient due to the idempotent sum and the fact that it is only needed one N' in the Taylor expansion of one summand such that $M \to_\varphi^* N'$ because the lemma holds. Take for instance $\mathbb{N}\sigma\vec{R} \triangleq N_1\sigma\vec{R} + N_2\sigma\vec{R} + N_3\sigma\vec{R}$ and say that $M \to_\varphi^* N' \in (N_1)^\circ$, while $M \to_\varphi^* 0$ in all other cases. Then, we would obtain a sum like $\mathbf{I}+0+0 \triangleq \mathbf{I}$. □

Let us remark some properties of the lemma 3.18 just proved. Two are the points we may stress. First of all, the sequence $\vec{R}$ of closed bags is independent from the sum $\mathbb{N}$; that is we can apply the same sequence $\vec{R}$ to all sums $\mathbb{L}$ satisfying a particular condition: M λ-wider than $\mathbb{L}$. The nice thing is that we can always make any term M λ-wider than any sum $\mathbb{N}$, as lemma 3.17 states. Here it comes the second feature to point out; that is, thanks to this condition we have for free the fact that we can distinguish M from an infinite set of terms. This is a huge difference from the pure λ-calculus case.

A resource conscious Böhm's theorem. We can now state the resource Böhm's theorem, but we need one more lemma before. It is a consistency lemma stating that the relation $\not\preceq_\tau$ is preserved under η-expansions.

Lemma 3.19. *Let* $\mathbb{M}, \mathbb{N} \in \Lambda^{r+}$. *If* $\mathbb{M} \not\preceq_\tau \mathbb{N}$ *then* $\mathbb{M}' \not\preceq_\tau \mathbb{N}$ *for all* $\mathbb{M}' \to_\eta^* \mathbb{M}$.

Proof. According with the contradiction proof technique, suppose there is a $\mathbb{M}' \to_\eta^* \mathbb{M}$ such that $\mathbb{M}' \preceq_\tau \mathbb{N}$. Then, by definition of $\preceq_\tau$, there are $\mathbb{M}'' \to_\eta^* \mathbb{M}'$ and $\mathbb{N}' \to_\eta^* \mathbb{N}$, such that $\mathbb{M}'' \sqsubseteq_\tau \mathbb{N}'$. Then, by transitivity of $\to_\eta^*$, also $\mathbb{M}'' \to_\eta^* \mathbb{M}$ holds. Hence, since $\mathbb{M}'' \to_\eta^* \mathbb{M}$ and $\mathbb{N}' \to_\eta^* \mathbb{N}$, by definition of $\preceq_\tau$, we get $\mathbb{M} \preceq_\tau \mathbb{N}$. Contradiction. □

Theorem 3.20 (Resource Böhm's theorem). *Let* $\mathbb{M}, \mathbb{N} \in \Lambda^{r+}$ *be closed sums in* NF_β. *If* $\mathbb{M} \not\equiv_{\tau\eta} \mathbb{N}$ *then there is a sequence* $\vec{P}$ *of closed bags such that either* $\mathbb{M}\vec{P} \to_\beta^* \mathbf{I}$ *and* $\mathbb{N}\vec{P} \to_\beta^* 0$, *or vice versa.*

Proof. From the hypothesis $\mathbb{M} \not\equiv_{\tau\eta} \mathbb{N}$, it holds either $\mathbb{M} \not\preceq_\tau \mathbb{N}$ or $\mathbb{N} \not\preceq_\tau \mathbb{M}$. Say $\mathbb{M} \not\preceq_\tau \mathbb{N}$. Applying lemma 3.19 we have $\mathbb{M}' \not\preceq_\tau \mathbb{N}$ for all $\mathbb{M}' \to_\eta^* \mathbb{M}$; in particular, by lemma 3.17, $\mathbb{M}' \not\preceq_\tau \mathbb{N}$ holds for a $\mathbb{M}'$ λ-wider than both $\mathbb{M}$ and $\mathbb{N}$. Lemma 3.14 tells us that there is $M' \in (\mathbb{M}')^\circ$ such that $\forall N \in \mathbb{N}^\circ$ we have $M' \not\to_\varphi^* N$. Note that such term M is β-normal since $\mathbb{M}'$ is β-normal and is λ-wider than both $\mathbb{M}$ and $\mathbb{N}$ by remark 3.16. We can now use the lemma 3.18, noticing that $M', \mathbb{M}, \mathbb{N}$ are closed. We can apply the lemma on M' and $\mathbb{M}$ obtaining a sequence $\vec{P}$ of closed bags such that $\mathbb{M}\vec{P} \to_\beta^* \mathbf{I}$, since $\mathbb{M}' \to_\eta^* \mathbb{M}$ and thus, by remark 3.12, there is a $M \in \mathbb{M}^\circ$ such that $M' \to_\varphi^* M$. Since M' is λ-wider than $\mathbb{N}$ too, we can use the same $\vec{P}$ obtaining $\mathbb{N}\vec{P} \to_\beta^* 0$, since for all $N \in \mathbb{N}^\circ$ we have $M' \not\to_\varphi^* N$. □

Notice here the difference from the separation theorem proved for the pure λ-calculus. In fact, in that case we can decide which of the two λ-terms delivers to $\mathbf{F}$ and which one to $\mathbf{T}$. Here we cannot, due to the fact that $0 \in \Lambda^{e+}$. As stated at the beginning of this sections, the obtained result is more similar to Hyland's semi-separation than Böhm's theorem because of the empty sum 0 who turns out to be an unsolvable β-normal form in $\Lambda^{\mathcal{R}}$, as proved in [12] by Pagani.

Like in the pure λ-calculus, we have that the $\tau\eta$-equivalence induces the maximal non-trivial congruence on β-normalizable terms extending the β-equivalence.

Corollary 3.21. *Let $\sim$ be a congruence on Λ^{e+} extending β-equivalence. If there are two closed $\mathbb{M}, \mathbb{N} \in \Lambda^{r+}$ in β-normal form such that $\mathbb{M} \not\equiv_{\tau\eta} \mathbb{N}$ but $\mathbb{M} \sim \mathbb{N}$, then is trivial, i.e. for all sums $\mathbb{L} \in \Lambda^{r+}$, $\mathbb{L} \sim 0$.*

Proof. Suppose $\mathbb{M} \not\equiv_{\tau\eta} \mathbb{N}$, but $\mathbb{M} \sim \mathbb{N}$. From theorem 3.20 there is $\vec{P}$ such that $\mathbb{M}\vec{P} \to_\beta^* \mathbf{I}$ and $\mathbb{N}\vec{P} \to_\beta^* 0$, or vice versa. Since $\sim$ is a congruence, $\mathbb{M} \sim \mathbb{N}$ entails $\mathbb{M}\vec{P} \sim \mathbb{N}\vec{P}$. Now, by the hypothesis that $\sim$ extends the β-equivalence, we get $\mathbf{I} \sim 0$. Thus, take any term $\mathbb{L} \in \Lambda^{r+}$, we have $\mathbb{L} \equiv_\beta \mathbf{I}\mathbb{L} \sim 0\mathbb{L} \triangleq 0$.[12] □

4 Conclusions

The work done can be divided into two related macro thematics. At first, we have presented the pure λ-calculus and its the original separation theorem, namely Böhm's theorem, investigating its semantic consequences and mentioning its several generalizations. Then, it has been shown a complete proof of Böhm's theorem following a technique due to Joly. It turns out that this last approach is, more or less, a different and efficient coding of the intuition behind the original proof introduced by Böhm.

In a second moment, we have introduced the resource λ-calculus, known also as differential λ-calculus à la Tranquilli. At the same time, we have given some intuitions on the links between this calculus and the differential λ-calculus. We have presented and discussed a particular operation of these calculi, namely the Taylor expansion. Finally, we focused on studying in depth and stressing details of the publication due to Manzonetto and Pagani on a formalization of Böhm's theorem for the resource λ-calculus. We have seen that this result is based on a syntactic characterization of the Taylor expansion operation and no more on the $\beta\eta$-equivalence. We have remarked that this theorem is more likely the one proved by Hyland in the pure λ-calculus setting. That is, we should indicate the result of Manzonetto and Pagani as a semi-separation theorem. As in pure setting, we have observed that the $\tau\eta$-equivalence induces the maximal non-trivial congruence on β-normalizable terms extending the β-equivalence.

As the reader should have understood, the semi-separation theorem studied in this report is not sufficient in a quantitative setting. This is obvious because the calculus considered is qualitative, i.e. it does not make any different between the sums, like $M + M$, and simple terms, like M. In this calculus, there are no coefficients related to terms. Thus, the theorem obtained cannot distinguish between two terms having different coefficients, say $3\mathbf{I}$ and $2\mathbf{I}$. Hence, the result obtained by Manzonetto and Pagani is not an answer to the separation problem in the sense of the very first aim of the internship.

As future works, we could begin to think about the meaning of a separation result in a quantitative setting, trying to get a Böhm's theorem in a quantitative calculus such as the algebraic λ-calculus proposed by Lionel Vaux. In this setting there is a clearer distinction between terms 0 and $\mathbf{\Omega}$, and coefficients do matter. It would be nice to investigate if it is possible to achieve a finer notion of separation and in which sense. Obviously, one possibility of research could be to extend the notion of Böhmtree to a quantitative setting.

Personal thoughts on the internship. As a foreign student approaching his first research experience, I consider the work done at LDP quite good, although there are no real contributions made on the subject of this report. If it can be considered a contribution, all the intuitions and explanations reported in this report are mine. Moreover, I tried to reorganize and make easier to follow the proof of lemma 3.18. Finally, all the examples written in the report are original. As stated before, I mostly focused on studying in details the various aspects of the separation problem in the pure λ-calculus and in the resource λ-calculus. In particular, this last calculus was completely new to me and I spent some time to understand it. I was unfamiliar either with the separation problem itself and original Böhm's theorem.

I found interesting to have a global picture on the subject, trying to read as much as possible from the literature. Indeed, I have studied notions and results that I did not write down in this report due to constraints on the maximal number of pages. For instance, in the context of the pure λ-calculus, I have faced a related problem to separation known as solvability. I also had a look at the same problematic in the resource setting. Moreover, for the pure λ-calculus I studied various versions of Böhm's theorem (i.e Böhm's, Krivine's and Joly's versions) and the generalization of the separation theorem to a finite set of normal λ-terms [4].

[12] Remember that 0 nullify every term or bag. In this specific case, $0P \triangleq 0$

While approaching the separation problem for the resource λ-calculus, I spent some effort to understand the links between this calculus and the differential λ-calculus, studying its origins and main ideas. Finally, I focused on studying in depth and stressing details of the publication due to Manzonetto and Pagani. It turns out to be a complicated job because of the strong technical nature of its presentation and development.

My personal judgment of this research experience is mostly motivated by the huge amount of information, notions, results and problems I have faced. For sure, I have a more complete idea on the separation problem and, more generally, on what is the meaning of doing research.

It remains some time before leaving Marseille and, for sure, I will continue working on the research directions written just before. In particular, I will investigate on a proof à la Joly for the lemma 3.18 and I will begin to think about how to approach the separation problem in a quantitative setting, since we cannot extend easily to it the technique studied here to. In fact, in this setting $\sqsubseteq_\tau$ is no more a preorder.

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A Böhm's theorem à la Joly: an example

We will consider the following two λ-terms $T_1 \triangleq xy(xxy)$ and $T_2 \triangleq xy(xzy)$. We will denote with $\star$ a λ-term whatever. Obviously, we will need to make two induction steps since the difference is due to x and z which are subterms of (xxy) and (xzy) respectively. These last two are themselves subterms of T_1 and T_2 respectively. We will indicate with T_{ij} the λ-term i at induction step j.

The base induction case, the one who really does separate the different subterms, is reported in details in figure 1.

λ-terms to separate:

- $T_{12} \triangleq x$
- $T_{22} \triangleq z$

Hence, we will have $\vec{Q} \triangleq \emptyset$ and $\vec{P} \triangleq P_x P_y P_z$. In particular, we will set $P_x \triangleq y_1$, $P_y \triangleq \star$ and $P_z \triangleq y_2$. Note that we applied the steps described in the very first "base case" detailed in the proof of theorem 2.15.

Figure 1: Induction base case

Once obtained the result from induction base case, we have to finish computing its upper level, namely the first induction step. The reader have the entire computation in figure 2.

λ-terms to separate:

- $T_{11} \triangleq (xxy)$
- $T_{21} \triangleq (xzy)$

Note that the number of parameters on the above λ-terms is $n = 2$. Now, we will use the lemma 2.14. Doing so we compute first $\vec{P'} \triangleq P'_x P'_y P'_z$, where $P'_x \triangleq \langle P_x, \lambda z_1 z_2.z_1 \rangle$, $P'_y \triangleq P_y$ and $P'_z \triangleq P_z$. At this level, since $\vec{Q} \triangleq \emptyset \Rightarrow p = 0$ we must choose $k \geq n + p \geq 2$. We will pick $k = 2$. Moreover, we obtain $l_{12} = l_{22} = l = 3$, $l'_{12} = 2$ and $l'_{22} = 3$. We obtain than:

- $T_{11}[x := P'_x; y := P'_y; z := P'_z]\vec{Q}\mathbf{ft}^3 \triangleq P'_x P'_x P'_y \mathbf{ft}^3 \to^*_\beta \langle P_x P'_x P'_y, \lambda z_1 z_2.z_1 P'_x P'_y \rangle_0 \mathbf{ft}^3 \to_\beta (\lambda z_1 z_2.z_1 P'_x P'_y)\mathbf{t}^3 \to_\beta P'_x \mathbf{t}^3 \to^*_\beta y_1 \mathbf{t}^2$
- $T_{21}[x := P'_x; y := P'_y; z := P'_z]\vec{Q}\mathbf{ft}^3 \triangleq P'_x P'_z P'_y \mathbf{ft}^3 \to^*_\beta \langle P_x P'_z P'_y, \lambda z_1 z_2.z_1 P'_z P'_y \rangle_0 \mathbf{ft}^3 \to_\beta (\lambda z_1 z_2.z_1 P'_z P'_y)\mathbf{t}^3 \to_\beta P'_z \mathbf{t}^3 \to^*_\beta y_2 \mathbf{t}^3$

Therefore, we can now compute $\vec{U}$ and $\vec{V}$ for this level of induction.

- $\vec{U} \triangleq \vec{P'}[y_i := \vec{\lambda z}_{l'_i}.y_i]$
- $\vec{V} \triangleq \mathbf{ft}^3$ (remember that $\vec{Q} \triangleq \emptyset$)

Figure 2: First induction step

It remains to compute at the top level using the information resulting from the first induction step. Following the notation of the proof, we will rename the last $\vec{U}, \vec{V}$ in $\vec{P}, \vec{Q}$ respectively. Last computations are presented in figure 3.

λ-terms to separate:

- $T_1 \triangleq xy(xxy)$
- $T_2 \triangleq xy(xzy)$

Note that the number of parameters on the above λ-terms is $n = 2$; moreover, we have $\vec{Q} \triangleq \mathbf{ft}^3$ from the first induction step implying $p = 4$. Now, we will use the lemma 2.14. Doing so we compute first $\vec{P'} \triangleq P'_x P'_y P'_z$, where $P'_x \triangleq \langle P_x \triangleq \langle \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2, \lambda z_1 z_2.z_2\rangle$, $P'_y \triangleq P_y$ and $P'_z \triangleq P_z \triangleq \lambda z_1 z_2 z_3.y_2$. Now, we must choose $k \geq n + p \geq 6$. We will pick $k = 6$. Moreover, using the lemma we obtain: $l_{11} = l_{21} = l = 6$, $l'_{11} = 4$ and $l'_{21} = 5$. Indeed, detailing the computation:

- $M_{11}[x := P'_x; y := P'_y; z := P'_z]\vec{Q}\mathbf{t}^6 \triangleq T_{11}[x := P'_x; y := P'_y; z := P'_z]\mathbf{ft}^3\mathbf{t}^6 \triangleq P'_x P'_x P'_y \mathbf{ft}^9 \to^*_\beta \langle \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2 P'_x P'_y \mathbf{ft}^8 \to^*_\beta P'_x \mathbf{t}^7 \to^*_\beta y_1 \mathbf{t}^4$
- $M_{21}[x := P'_x; y := P'_y; z := P'_z]\vec{Q}\mathbf{t}^6 \triangleq T_{21}[x := P'_x; y := P'_y; z := P'_z]\mathbf{ft}^3\mathbf{t}^6 \triangleq P'_x P'_z P'_y \mathbf{ft}^9 \to^*_\beta \langle \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2 P'_z P'_y \mathbf{ft}^8 \to^*_\beta P'_z \mathbf{t}^8 \to^*_\beta y_2 \mathbf{t}^5$

We obtain than:

- $T_1[x := P'_x; y := P'_y; z := P'_z]\vec{Q}\mathbf{ft}^6 \triangleq P'_x P'_y (P'_x P'_x P'_y)\mathbf{ft}^3\mathbf{ft}^6 \to^*_\beta \langle\langle \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2 P'_y (P'_x P'_x P'_y)\mathbf{ft}^3, \lambda z_1 z_2.z_2 P'_y (P'_x P'_x P'_y)\mathbf{ft}^3\rangle_0 \mathbf{ft}^6 \to^*_\beta P'_x P'_x P'_y \mathbf{ft}^9 \to^*_\beta \langle \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2 P'_x P'_y \mathbf{ft}^8 \to^*_\beta P'_x \mathbf{t}^7 \to^*_\beta y_1 \mathbf{t}^4$
- $T_2[x := P'_x; y := P'_y; z := P'_z]\vec{Q}\mathbf{ft}^6 \triangleq P'_x P'_y (P'_x P'_z P'_y)\mathbf{ft}^3\mathbf{ft}^6 \to^*_\beta \langle\langle \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2 P'_y (P'_x P'_z P'_y)\mathbf{ft}^3, \lambda z_1 z_2.z_2 P'_y (P'_x P'_z P'_y)\mathbf{ft}^3\rangle_0 \mathbf{ft}^6 \to^*_\beta P'_x P'_z P'_y \mathbf{ft}^9 \to^*_\beta \langle \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2 P'_z P'_y \mathbf{ft}^8 \to^*_\beta P'_z \mathbf{t}^8 \to^*_\beta y_2 \mathbf{t}^5$

Therefore, we can now conclude computing the final $\vec{U}$ and $\vec{V}$.

- $\vec{U} \triangleq \vec{P'}[y_1 := \lambda\vec{w}_4.y_1; y_2 := \lambda\vec{w}_5.y_2]$. That is:
  - $U_x \triangleq P'_x[y_1 := \lambda\vec{z}_4.y_1; y_2 := \lambda\vec{z}_5.y_2] \triangleq \langle\langle \lambda w_1 \ldots w_4 \lambda z_1 z_2.y_1, \lambda z_1 z_2.z_1\rangle_2, \lambda z_1 z_2.z_2\rangle_6$
  - $U_y \triangleq P'_y \triangleq \star$
  - $U_z \triangleq P'_z[y_1 := \lambda\vec{w}_4.y_1; y_2 := \lambda\vec{w}_5.y_2] \triangleq \lambda w_1 \ldots w_5 \lambda z_1 z_2 z_3.y_2$
- $\vec{V} \triangleq \mathbf{ft}^3\mathbf{ft}^6$ (remember that $\vec{Q} \triangleq \mathbf{ft}^3$)

Figure 3: Top level

B Resource Böhm's theorem: an example

We will consider the following two resource λ-terms $U \triangleq \lambda x.x$ and $\mathbb{W} \triangleq \lambda x\lambda y.x[y^2] + \lambda x.x[x^!]$. Obviously, $U \not\equiv_{\tau\eta} \mathbb{W}$ since they have different Taylor expansions and they cannot reduce to each other by η-reductions.

Following what theorem 3.20 says, we need to η-expand the term U because it must be λ-wider than every summand of $\mathbb{W}$. Take for example $Z \triangleq \lambda x\lambda y.x[y^!]$ noticing that $Z \to_\eta U$ and $Z \not\to_\eta \mathbb{W}$. From lemma 3.19 and lemma 3.14, we know that there is $Z' \in Z^\circ$ such that for all $W \in \mathbb{W}^\circ$ it holds $Z' \not\to_\varphi^* W$. In our specific case, we can pick $Z' \triangleq \lambda x\lambda y.x[y]$.

Let us now use the lemma 3.18 with our terms Z' and U act, respectively, as terms M and $\mathbb{N}$ who are in the statement of the lemma.

We will need a step of induction since Z' has got a bag of resources. The reader can follow the entire computation reported in the figures below.

Resource λ-terms we are working on:

- $Z' \triangleq \lambda x\lambda y.x[y]$ acting as M
- $U \triangleq \lambda x.x$ acting as $\mathbb{N}$

Notice in the proof of theorem 3.20 that the amount to assign to k is determined at the top level of the induction; i.e. $k > size(Z') = 6$. We pick $k = 7$. Moreover, we have:

- number of bags: $p = 1$
- head variable: $x_h = x$
- the substitution σ is empty since Z' and U are closed

Figure 4: Top level case

Let us make the induction step on the bag $[y]$, in figure 5. We will resume the top level computation in figure 6.

At this level of computation, our term M is y. Obviously, we have:

- number of bags: $p_1 = 0$
- head variable: $x_h = y$

Since x, y are free at this point, we must set:

- $k_y > 2k = 12$. We pick $k_y = 15$
- $k_x > 2k$ and $|k_x - k_y| > 2k$. We pick $k_x = 30$.

Thus, we get $m_1 = max\{k_y, k_x\} = 30$. The term H_1 we obtain is the following: $H_1 \triangleq \lambda \vec{w}_{30}.\mathbf{I}$. Hence, we have as sequence of closed bags $\vec{R_1}$ as: $\vec{R_1} \triangleq 1^{15}[H_1^!]1^{15}$.

Figure 5: Induction base case

It remains to finish the computation of the top level using the results just obtained above. The reader can follow this steps in figure 6.
Finally, in figure 7 it is reported complete computations resulting from applying U and $\mathbb{W}$ to $\vec{R}$.

Resource λ-terms we are working on:

- $Z' \triangleq \lambda x \lambda y.x[y]$ acting as M
- $U \triangleq \lambda x.x$ acting as $\mathbb{N}$

Having the results from the induction step, $\vec{R_1}$, we first define the term H as:

$$H \triangleq \lambda z_1 \lambda \vec{w}_{59}.\mathbf{I}[z_1 \vec{R_1}]$$

Finally, we build the sequence of closed bags $\vec{R}$ as:

$$\vec{R} \triangleq [\mathbf{rP}^!_{k_x}][\mathbf{rP}^!_{k_y}]1^{29}[H^!]1^{30}$$

where k_x, k_y are those defined in the induction base case. Moreover, remember the structure of the *resource n-permutator*:

$$\mathbf{rP}_n \triangleq \lambda \vec{x}_n \lambda x.x[x^!_1] \dots [x^!_n]$$

Figure 6: Top level case (completed)

- $U\vec{R} \triangleq (\lambda x.x)[\mathbf{rP}^!_{k_x}][\mathbf{rP}^!_{k_y}]1^{29}[H^!]1^{30} \to_\beta$ $(\mathbf{rP}_{k_x})[\mathbf{rP}^!_{k_y}]1^{29}[H^!]1^{30} \triangleq (\lambda x_1 \dots \lambda x_{30} \lambda x.x[x^!_1] \dots [x^!_{30}])[\mathbf{rP}^!_{k_y}]1^{29}[H^!]1^{30} \to^*_\beta H[\mathbf{rP}^!_{k_y}]1^{59} \triangleq \lambda z_1 \lambda \vec{w}_{59}.\mathbf{I}[z_1 R_1] \to^*_\beta \mathbf{I}[\mathbf{rP}_{k_y} \vec{R_1}] \triangleq \mathbf{I}[(\lambda \vec{y}_{15} \lambda x.x[x^!_1] \dots [x^!_{15}])1^{15}[H^!_1]1^{15}] \to^*_\beta \mathbf{I}[H_1 1^{30}] \triangleq \mathbf{I}[(\lambda \vec{w}_{30}.\mathbf{I})1^{30}] \to^*_\beta \mathbf{I}[\mathbf{I}] \to_\beta \mathbf{I}$

- $\mathbb{W}\vec{R} \triangleq (\lambda x \lambda y.x[y^2])\vec{R} + (\lambda x.x[x^!])\vec{R}$. We will handle each summand on its own and sums the result at the end.
  - $(\lambda x \lambda y.x[y^2])\vec{R} \to^*_\beta \mathbf{rP}_{k_x}[\mathbf{rP}_{k_y}, \mathbf{rP}_{k_y}]1^{29}[H^!]1^{30} \to^*_\beta$ $H[\mathbf{rP}_{k_y}, \mathbf{rP}_{k_y}, x^!_1]1^{59} \triangleq (\lambda z_1 \lambda \vec{w}_{59}.\mathbf{I}[z_1 \vec{R_1}])[\mathbf{rP}_{k_y}, \mathbf{rP}_{k_y}, x^!_1]1^{59} \to_\beta 0$
  - $(\lambda x.x[x^!])\vec{R} \to_\beta$ $(\mathbf{rP}_{k_x}[\mathbf{rP}^!_{k_x}])[\mathbf{rP}^!_{k_y}]1^{29}[H^!]1^{30} \triangleq (\lambda \vec{x}_{30} \lambda x.x[x^!_1] \dots [x^!_{30}])[\mathbf{rP}^!_{k_x}][\mathbf{rP}^!_{k_y}]1^{29}[H^!]1^{30} \to^*_\beta$ $(\lambda x.x[\mathbf{rP}^!_{k_x}][\mathbf{rP}^!_{k_y}]1^{28})1[H^!]1^{30} \to_\beta 0$

  Thus, we have: $0 + 0 = 0$.

Summarizing, we obtain what we wanted: $U\vec{R} \to^*_\beta \mathbf{I}$ and $\mathbb{W}\vec{R} \to^*_\beta 0$.

Figure 7: Applying $U, \mathbb{W}$ to the $\vec{R}$ just built